Guarding Polyominoes under k-Hop Visibility or Minimum k-Dominating Sets in Grid Graphs*

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Abstract

We consider a guarding problem in polyominoes: a unit-square guard sees all unit squares within distance k in the dual grid graph. The problem is equivalent to the minimum k-dominating set problem in grid graphs. We prove NP-completeness of this problem in polyominoes with holes for $k \in \{1, 2\}$ and provide lower bounds for all k and matching upper bounds for $k \in \{1, 2\}$ of $\lfloor \frac{m}{k+1} \rfloor$ on the number of guards in any polyomino.

1 Introduction

In the classical art gallery problem (AGP), we aim to place guards in a polygon, such that every point of the polygon is visible to at least one guard. Visibility is defined by analogy to human vision: two points $u, v \in P$ see each other if the line segment $\overline{u, v}$ is fully contained in P. Various variants for the classical AGP have been considered (varying both the capabilities of the guards and the environment to be guarded), and usually we are interested in two types of questions:

1. Can we compute the minimum cardinality guard set for a polygon P?
2. What are lower/upper bounds on the number of guards needed to cover a polygon from a given class?

For the classical AGP, question (1) was answered with several complexity results: NP-hardness was proven for different problem variants ([1, 2]). Answers to question (2) are often referred to as "Art Gallery theorems". Chvátal [3] provided the first such result: a tight bound of $\lfloor \frac{n}{3} \rfloor$ for simple polygons.

Here, we consider a guarding problem motivated from serving a city with carsharing (CS) stations: the demand is given in a granularity of (square) cells, and we assume that customers are willing to walk a certain distance to a CS stations—a simplified assumption, which we can substitute by obtaining demand for given stations using a multi-agent transport simulation, MATSim[1]. We also assume that this walking-range bound is the same for the complete city. Then, we aim to place as few CS stations as possible to serve the complete city for a given maximum walking range. We represent the city as a polyomino, potentially with holes and only walking within the boundary is possible—e.g., in Stockholm holes usually represent water bodies, which pedestrians cannot cross. This yields a special type of "visibility" for a station: all unit squares of the polyomino reachable when walking inside the polyomino for at most the given walking range.

Guarding polyominoes has been considered by Biedl et al. [4], who considered different models of visibility. ($u \in P$ sees $v \in P$ if the axis-parallel rectangle spanned by u, v is fully contained in P). They provided both NP-hardness results and art gallery theorems in terms of the number of unit squares of the polyomino, m. NP-hardness for another type of visibility (rectangle visibility) was provided in [5].

An equivalent formulation of our problem is in terms of the minimum k-dominating set problem (MkDSP): find a minimum cardinality $D_k \subseteq V(G)$, such that each graph vertex is connected to a vertex in D_k with a path of length at most k. We aim to solve MkDSP in grid graphs (the dual graph of a polyomino). The minimum dominating set problem is NP-complete [6], hence, the MkDSP is clearly NP-complete in general graphs.

Notation. A *polyomino* is a connected polygon P in the plane formed by joining together $|P| = m$ unit squares on the square lattice. The dual graph G_P of a polyomino has a vertex for each unit square and $\{u, v\} \in E(G_P)$ if unit squares u, v are adjacent; G_P is a *grid graph*. P is *simple* if it has no holes, that is, if every minimal cycle in the dual grid graph is a 4-cycle.

A unit square $v \in P$ is *k-hop-visible* to a unit square $u \in P$ if the shortest path from u to v in the dual grid graph of P, G_P, has length at most k. See Figure 1 for an example. Note that for $k \geq 2$ this in particular includes the ability to look around a corner of the polyomino. A *witness* placed at unit square u vouches that at least one *guard* has to be placed in its k-hop-visibility region.

Minimum k-hop Guarding Problem (MkGP).
Given: A polyomino P, a range k.
Find: The minimum number cardinality unit-square guard cover in P under k-hop visibility.

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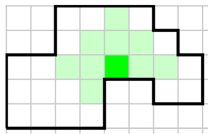


Figure 1: A polyomino P (black), a unit-square guard g (green) and its visibility region (light green), $k = 2$.

2 NP-Completeness

We show NP-completeness of the MkGP in polyominoes with holes and $k = 2$. We reduce from PLANAR 3-SAT.

An instance F of the PLANAR 3-SAT problem is a Boolean formula in 3-CNF consisting of a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of m clauses over n variables $\mathcal{V} = \{x_1, x_2, \ldots, x_n\}$. Clauses in F contain variables and negated variables, denoted as *literals*. A clause is satisfied iff it contains at least one true literal, and the formula F is true iff all its clauses are satisfied. The variable-clause incidence graph G is planar and it is sufficient to consider formulae where G has a rectilinear embedding, see Knuth and Raghunathan [7].

We turn the rectilinear embedding of G into a polyomino: we represent the variables, clauses and edges by pieces of a polyomino that needs to be guarded. We construct **variable gadgets** as shown in turquoise in Figure 2(a). There exist exactly two feasible placements of guards for the variable loop gadget, shown in blue and red in Figure 2(b) and (c) and corresponding to a truth setting of "true" and "false", respectively.

The initial truth value is propagated by a **wire gadget**. In Figure 2 we show a wire gadget for the case that the variable appears in a clause in dark blue, and for the case that the negated variable appears in the clause in dark red. We note that a wire can easily be bend by 90°. We extend the width of the variable gadget to connect to further wire gadgets.

The **clause gadget** is shown in Figure 3(a): Wires connecting to the three variables connect to it from the top, the right, and the bottom. The clause gadget can be covered with exactly two additional guards if one (see Figure 3(d)-(f)), two (see Figure 3(g)-(i)) or all (see Figure 3(j)/(k)) variables have a truth setting fulfilling the clause. If all variables have a truth setting not fulfilling the clause (see Figure 3(b)), three additional guards are needed to cover the clause gadget: The k-hop visibility regions of the three colored witnesses in Figure 3(b) are pairwise disjoint, hence, at least three guards are necessary—and sufficient, see Figure 3(c).

Thus, we solve the MkGP optimally iff 1-3 variables per clause have a truth setting fulfilling the clause, that is, iff the original PLANAR 3-SAT formula F is satisfiable. The reduction is possible in polynomial time. Moreover, given a set of unit-square guards, we can easily determine the k-hop visibility region of all guards and check whether all unit squares are covered. Hence, the MkGP is in NP. This yields:

Theorem 1 *MkGP is NP-complete for $k = 2$ in polyominoes with holes.*

A similar variable and corridor gadget construction and the clause from Fig. 4 yield NP-completeness also for $k = 1$ (due to space restrictions without proof):

Theorem 2 *MkGP is NP-complete for $k = 1$ in polyominoes with holes.*

3 Art Gallery Theorems

In this section, we provide lower bounds for all k (Theorem 3) and matching upper bounds for $k \in \{1, 2\}$ (Theorem 4) on the number of guards necessary to cover polyominoes under k-hop visibility.

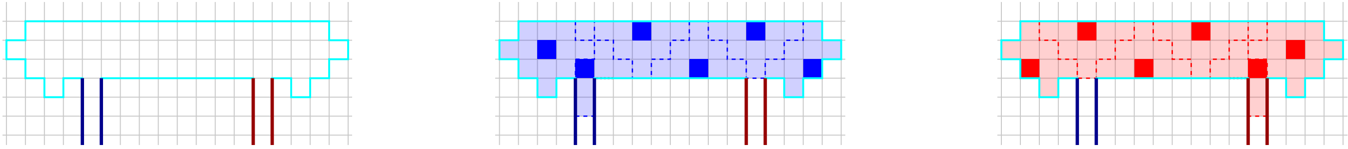
Theorem 3 *There exist simple polyominoes with m unit squares that require $\lfloor \frac{m}{k+1} \rfloor$ guards to cover their interior under k-hop visibility.*

Proof. We construct a double-comb like polyomino: we alternately add teeth of length k to the top and bottom of a row of unit squares (the shaft), see Figure 6 for the construction for $k = 1$ and $k = 2$. If m is not divisible by $k + 1$ we add $x = (m \mod k + 1)$ unit squares to the right of the shaft. Witnesses placed at the last unit square of each tooth (shown in pink in Figure 6) have disjoint k-hop visibility regions (of the shaft only the unit square to which the tooth is attached belongs to the k-hop visibility region), hence, we need one guard per witness. The x unit squares to the right of the shaft can be covered by the rightmost guard if placed in the shaft. Let t be the number of teeth, $m = t \cdot (k + 1) + x$, we need $t = \lfloor \frac{m}{k+1} \rfloor$ guards. □

Theorem 4 *$\lfloor \frac{m}{k+1} \rfloor$ guards are always sufficient and sometimes necessary to cover a polyomino with m unit squares under k-hop visibility for $k \in \{1, 2\}$.*

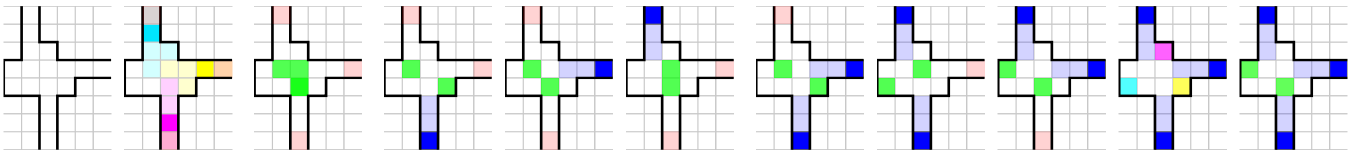
Proof. We need to show that $\lfloor \frac{m}{r+1} \rfloor$ guards are always sufficient. We give constructive proofs for $r \in \{1, 2\}$.

Case $k = 1$. Compute a maximum matching M in the (bipartite) dual grid graph of P, G_P. Every vertex in G_P that is not matched is adjacent to matched vertices only (otherwise we could extend M). For each matching edge $\{u, v\}$ unmatched vertices are adjacent to u or v only (otherwise, let u' and v' be an unmatched vertex adjacent to u and v, respectively, then $M \setminus \{u, v\} \cup \{u', u\} \cup \{v, v'\}$ is a larger matching than M). For each matching edge, we place a guard at the unit square of the vertex in G_P that is adjacent to unmatched



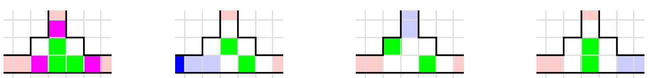
(a) (b) (c)

Figure 2: (a) Variable gadget in turquoise, wire gadgets in dark blue (in case the variable appears in a clause) and dark red (in case the negated variable appears in a clause). We associate the solution in (b) and (c) with a truth setting of "true" and "false", respectively.



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k)

Figure 3: (a) Clause gadget. (b)-(k) The truth setting of the variables connected by the three variable corridors is shown in red/blue, where light-blue/light-red indicates the visibility region of a guard, a red/blue unit square indicates the guard's location. (b) All variables have a truth setting that does not fulfill the clause, then the three colored witnesses (visibility regions in lighter colors) cannot be covered by the same guard, hence, three guards are necessary, and sufficient (c). If one variable has a truth setting that does fulfill the clause, (d)-(f), and if two variables have a truth setting that does fulfill the clause, (g)-(i), two (green) guards suffice to cover the clause gadget. If all variables have a truth setting that does fulfill the clause, the three witnesses in (j) cannot all be covered with a single guard, the two green guards in (k) are sufficient to cover the clause gadget.



(a) (b) (c) (d)

Figure 4: Clause gadget for $k = 1$. (a) If all variables have a truth setting not fulfilling the clause, the three pink witnesses cannot be covered by the same guards; three green guards are sufficient. (b) If one variable has a truth setting that fulfills the clause two green guards suffice. The other cases are omitted. Color usage as in Figure 3.

vertices (if any, otherwise we choose one of the two vertices). This guard covers its matched neighbor and all unmatched neighbors. Hence, we placed at most $\lfloor \frac{m}{k+1} \rfloor = \lfloor \frac{m}{2} \rfloor$ guards to cover P.

Case $k = 2$**.** Again, we compute a maximum matching M in G_P. We build a graph G_M based on M: we create a vertex in G_M for each matching edge and each unmatched vertex in M ($V(G_M) = \{v_{\{u,u'\}} : \{u,u'\} \in M\} \cup \{v_u : u \in G_P \setminus M\}$). Two vertices v, v' in G_M are connected by an edge if:

- For $v = v_{\{u,u'\}}, v' = v'_{\{w,w'\}}$: if at least one of the edges $\{u,w\}, \{u,w'\}, \{u',w\}$ or $\{u',w'\}$ is in $E(G_P)$.
- For $v = v_{\{u,u'\}}, v' = v'_w$: if at least one of $\{u,w\}$ or $\{u,w'\}$ is in $E(G_P)$.

We compute a maximum matching M' in G_M. Each matching edge in M' represents three or four vertices of G_P. Each unmatched vertex in M' represents one or two vertices of G_P. Again, unmatched vertices are adjacent to at most one of the vertices per edge in M'. If an unmatched vertex is adjacent to more than one matched vertex, we assign it to one of them. In the remainder of this proof adjacent unmatched vertex/vertices refers to the *assigned* adjacent unmatched vertex/vertices only. We distinguish six cases, see Figure 5 for examples:

1. $e = \{v_{\{u,u'\}}, v_{\{w,w'\}}\} \in M'$ is not adjacent to any unmatched vertex in M': we have a path of length 4, and place a guard on one of the two vertices that are adjacent to two of the other three vertices. Hence, the single guard covers 4 unit squares.
2. $e = \{v_{\{u,u'\}}, v_{\{w,w'\}}\} \in M'$ is adjacent to unmatched vertices, all unmatched vertices adjacent to e represent one vertex from G_P. W.l.o.g. let the unmatched vertices be adjacent to $v_{\{u,u'\}}$:
   (a) $\{u,w\} \in G_P$ or $\{u,w'\} \in G_P$, but $\{u',w\} \notin G_P$ and $\{u',w'\} \notin G_P$: We place a guard on u, then u, u', w, w' and all unmatched vertices adjacent to $v_{\{u,u'\}}$ are covered. The single guard at u covers at least 5 unit squares.
   (b) $\{u,w\} \in G_P$ and $\{u',w'\} \in G_P$ (or $\{u',w\} \in G_P$ and $\{u,w'\} \in G_P$): A guard placed on u or u' covers u, u', w, w' and all unmatched vertices adjacent to $v_{\{u,u'\}}$. The single guard covers at least 5 unit squares.
3. $e = \{v_{\{u,u'\}}, v_{\{w,w'\}}\} \in M'$ is adjacent to unmatched vertices, some unmatched vertices adjacent to e represent two vertices from G_P. Let the unmatched vertices be adjacent to $v_{\{u,u'\}}$: We

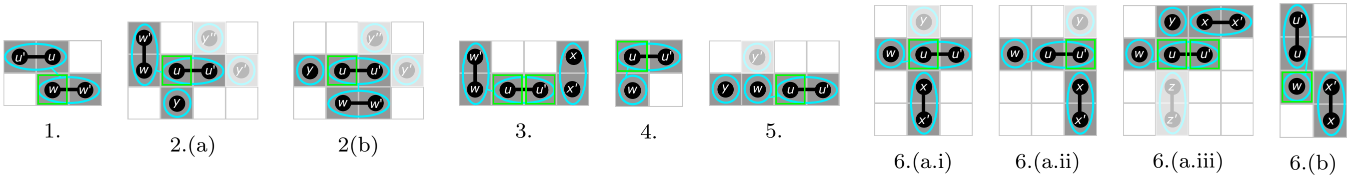


Figure 5: Cases from the proof of Lemma 4, $k = 2$. Vertices and M in G_P shown in black; vertices and M' in G_M shown in turquoise; guards shown in green. Optional unit squares are shown faded.

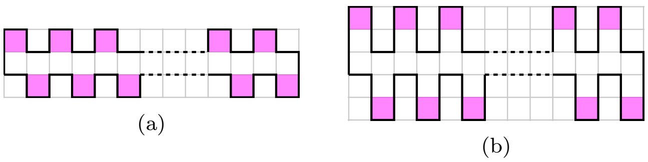


Figure 6: Lower bound construction for polyominoes that require $\lfloor \frac{m}{k+1} \rfloor$ guards under k-hop visibility for (a) $k = 1$, (b) $k = 2$. Witnesses are shown in pink.

place two guards at u and u' and cover u, u', w, w' and all unmatched vertices adjacent to $v_{\{u,u'\}}$. These two guards cover at least 6 unit squares: u, u', w, w' and at least one pair of vertices x, x', where $v_{\{x,x'\}}$ unmatched in M' and adjacent to $v_{\{u,u'\}}$. Hence, on average, each of the guards covers at least 3 unit squares.

4. $e = \{v_{\{u,u'\}}, v_w\} \in M'$ is not adjacent to any unmatched vertex in M', w.l.o.g. $\{u, w\} \in E(G_P)$: We place a guard on u, it covers 3 unit squares.
5. $e = \{v_{\{u,u'\}}, v_w\} \in M'$ is adjacent to unmatched vertices adjacent to e representing one vertex from G_P, w.l.o.g. $\{u, w\} \in E(G_P)$: We place a guard on u, which covers u, u' and w and all vertices adjacent to these three (independent of whether all are adjacent to $v_{\{u,u'\}}$ or to v_w). A single guard covers at least 4 unit squares.
6. $e = \{v_{\{u,u'\}}, v_w\} \in M'$ is adjacent to unmatched vertices, some of these represent two vertices from G_P, w.l.o.g. $\{u, w\} \in E(G_P)$:
   (a) The unmatched vertices are adjacent to $v_{\{u,u'\}}$:
      (i) For all unmatched vertices v_y, y is adjacent to u and for all unmatched vertices $v_{\{x,x'\}}$ x or x' is adjacent to u: We place a guard at u. This single guard covers at least 5 unit squares.
      (ii) For all unmatched vertices v_y, y is adjacent to u' and for all unmatched vertices $v_{\{x,x'\}}$ x or x' is adjacent to u': We place a guard at u'. This single guard covers at least 5 unit squares.
      (iii) We have at least one unmatched vertex for which one of the vertices in G_P it represents is adjacent to u and one for which one of the vertices in G_P it represents is adjacent to u'. We place two guards at u and u', all adjacent unmatched vertices representing a matching edge contain vertices within distance at most 2 from u and u'. The two guards cover at least u, u', w, at least one pair of vertices x, x', where $v_{\{x,x'\}}$ unmatched in M' and adjacent to $v_{\{u,u'\}}$ and least another single vertex or vertex pair adjacent to $v_{\{u,u'\}}$. Hence, 2 guards cover at least 6 unit squares—on average, each guard covers at least 3 unit squares.
   (b) The unmatched vertex/vertices are adjacent to v_w: We place a guard at w; it sees u, u', w as well as at least one pair of vertices x, x', where $v_{\{x,x'\}}$ unmatched in M' and adjacent to v_w because all these vertices have distance at most 2 to w. The guard covers at least 5 unit squares.

Each guard covers at least 3 unit squares, hence, we yield that $\lfloor \frac{m}{k+1} \rfloor = \lfloor \frac{m}{3} \rfloor$ guards are always sufficient. $\square$

4 Open Problems

We leave the computational complexity in simple polyominoes and upper bounds on the number of guards necessary to cover polyominoes under k-hop visibility for $k \geq 3$ as open problems.

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