Guarding Polyominoes under k-hop Visibility or Minimum k-Dominating Sets in Grid Graphs

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Agenda

- Motivation and Formal Problem Definition
- NP-Completeness
- Art Gallery Theorems
- Open Problems

Motivation

- Serve a city with carsharing (CS) stations:
  - Demand in granularity of square cells
  - Customers willing to walk a certain distance
  - Same distance bound for the complete city
  - City ➜ Polyomino
  - Walking only within the polyomino
- Goal: Place as few CS stations as possible to serve the complete city

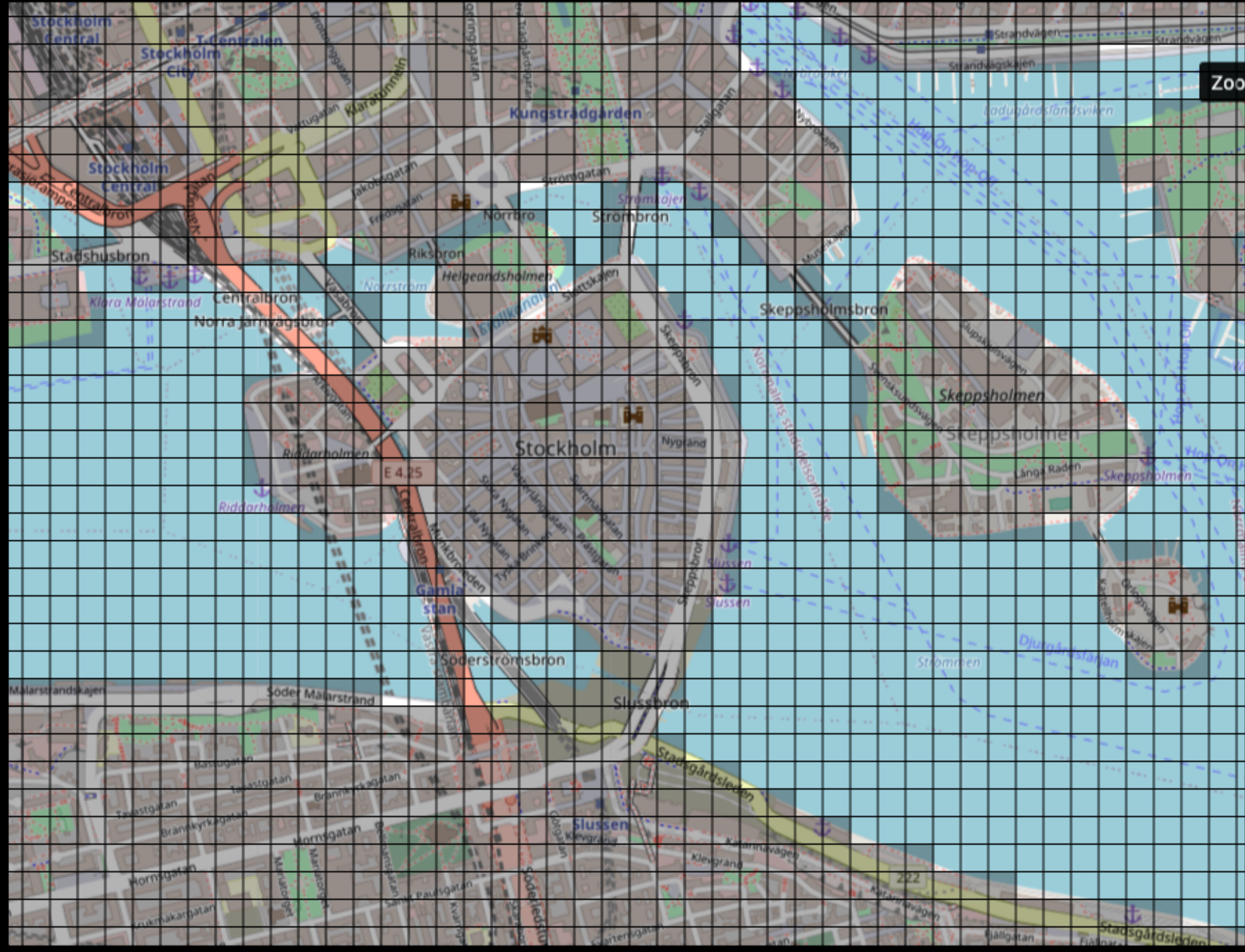


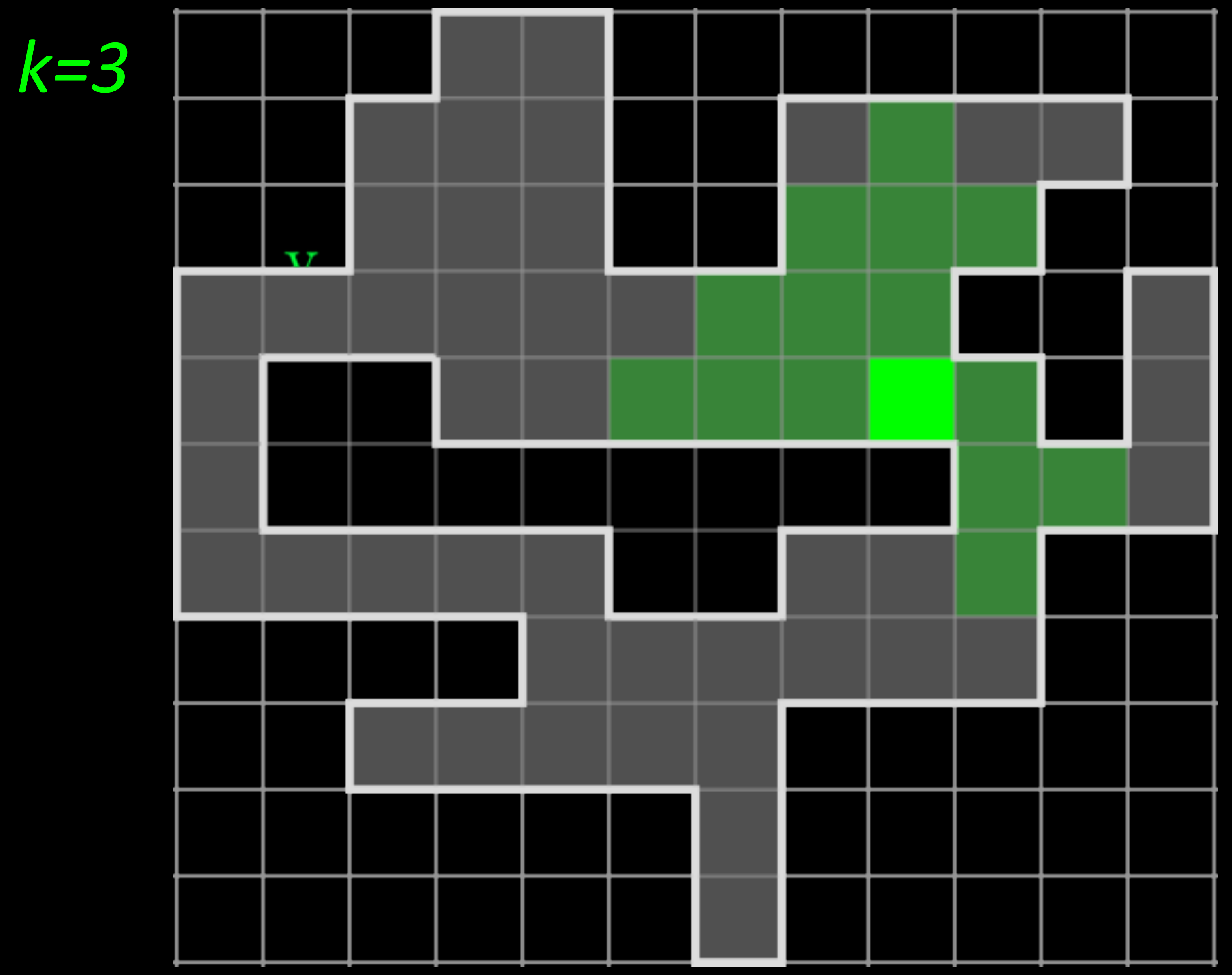
Image source: Gunnar Flötteröd, Waterborne Urban Mobility, Final Project Report

Motivation

- So, what can a station serve?
  - All unit squares of the polyomino reachable when walking when walking inside the polyomino for at most the given walking range
  - Walking range k
  - "Visibility": We can look around corners for $k \geq 2$

k=5

Formal Definition



- Polyomino: connected polygon P in plane, formed by joining m unit squares on the square lattice
- Dual graph G_P is a grid graph
- Unit square $v \in P$ *k-hop visible* to unit square $u \in P$, if shortest path from u to v in G_P has length at most k.

Minimum *k*-hop Guarding Problem (M*k*GP)

Given: Polyomino P, range k

Find: Minimum cardinality unit-square guard cover in P under k-hop visibility.

Alternative Formulation

Minimum *k*-dominating Set Problem (M*k*DSP)

Given: Graph G

Find: Minimum cardinality $D_k \subseteq V(G)$, each graph vertex connected to vertex in D_k with a path of length at most k.

M*k*DSP is NP-complete in general graphs.

➡ We want to solve M*k*DSP in grid graphs

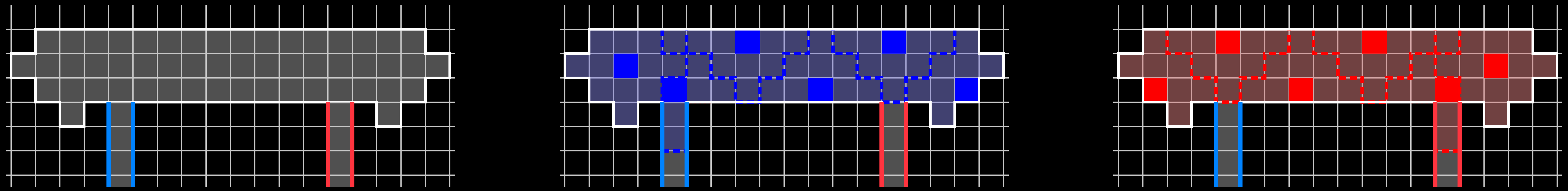
NP-Completeness

Theorem 1: MkGP is NP-complete for k=2 in polyominoes with holes.

- Proof by reduction from PLANAR 3 SAT
- Given a set of guards it can be verified in polynomial time whether each unit square of the polyomino is covered

Variable gadget with two corridor gadgets:

- In case variable appears in clause
- In case negated variable appears in clause

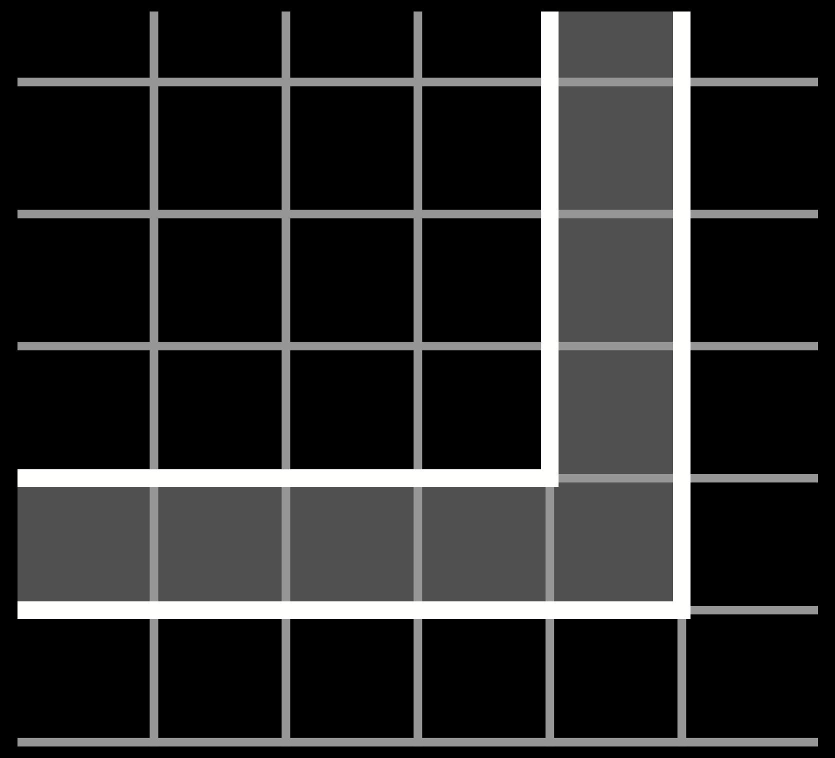


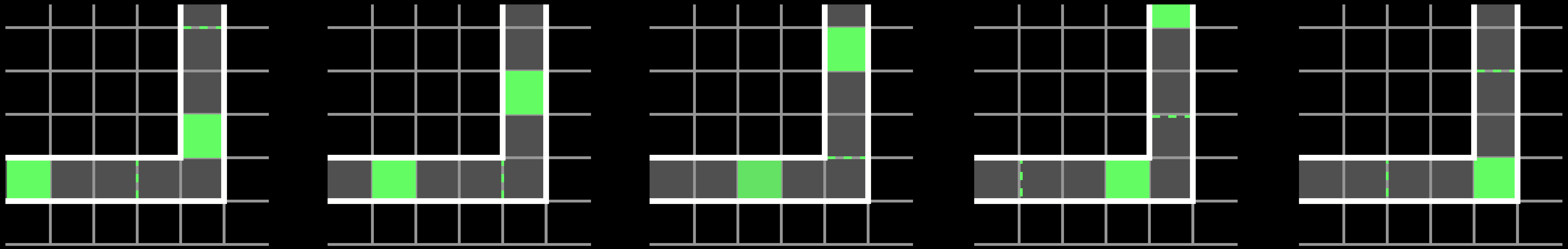
Corridor propagates variable value

“true”

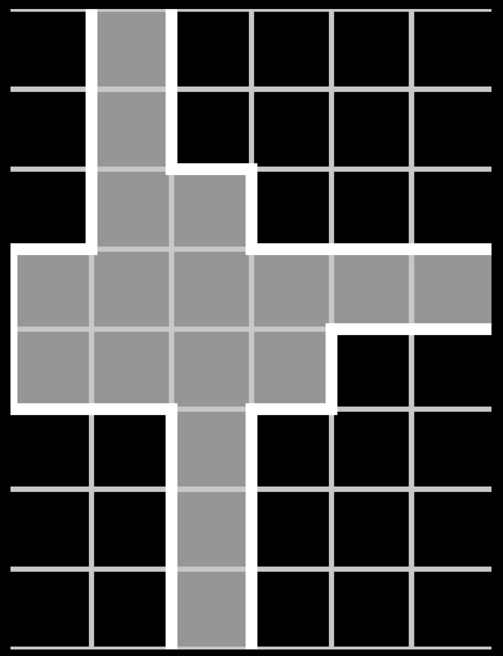
“false”

Corridor bend:

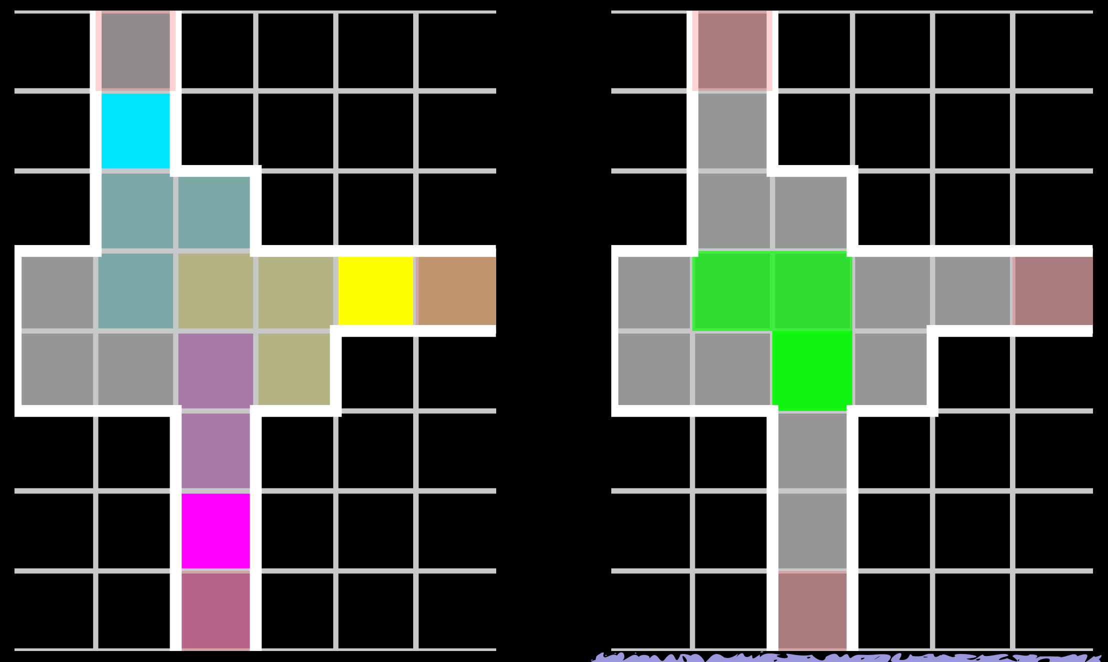




Clause gadget

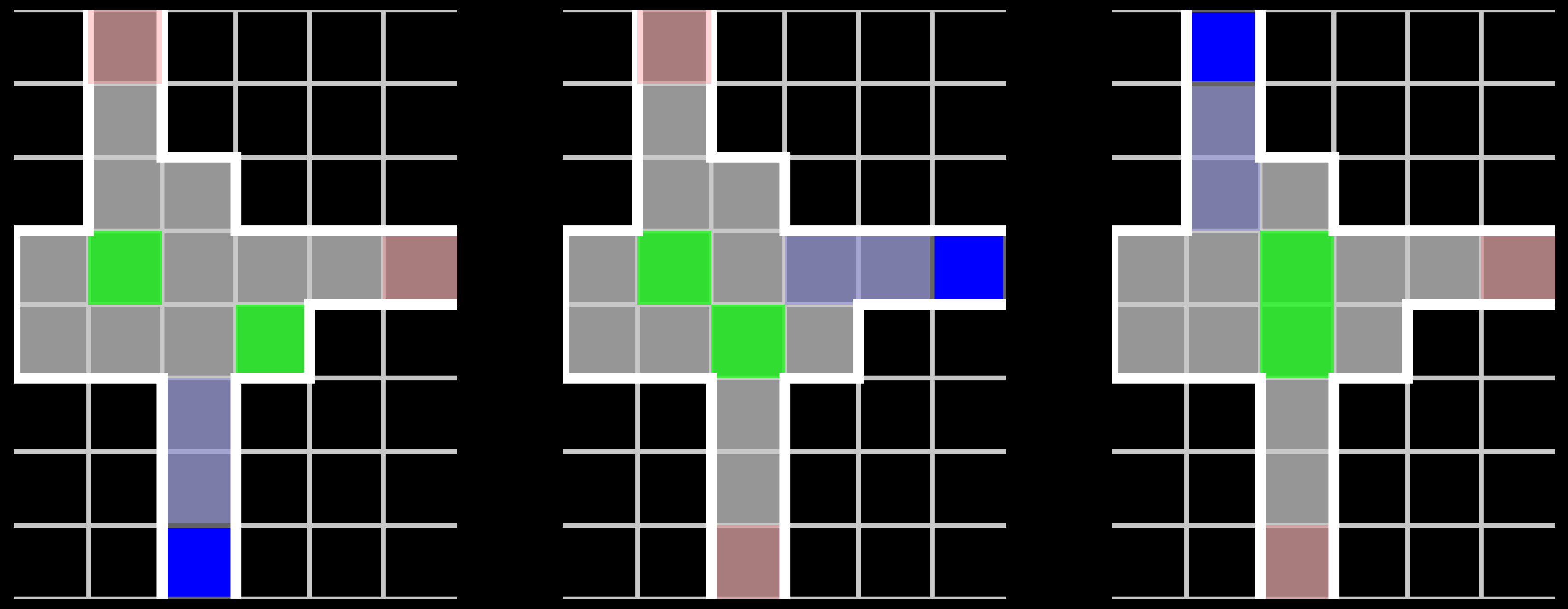


FFF: 3 witnesses - pairwise disjoint visibility regions



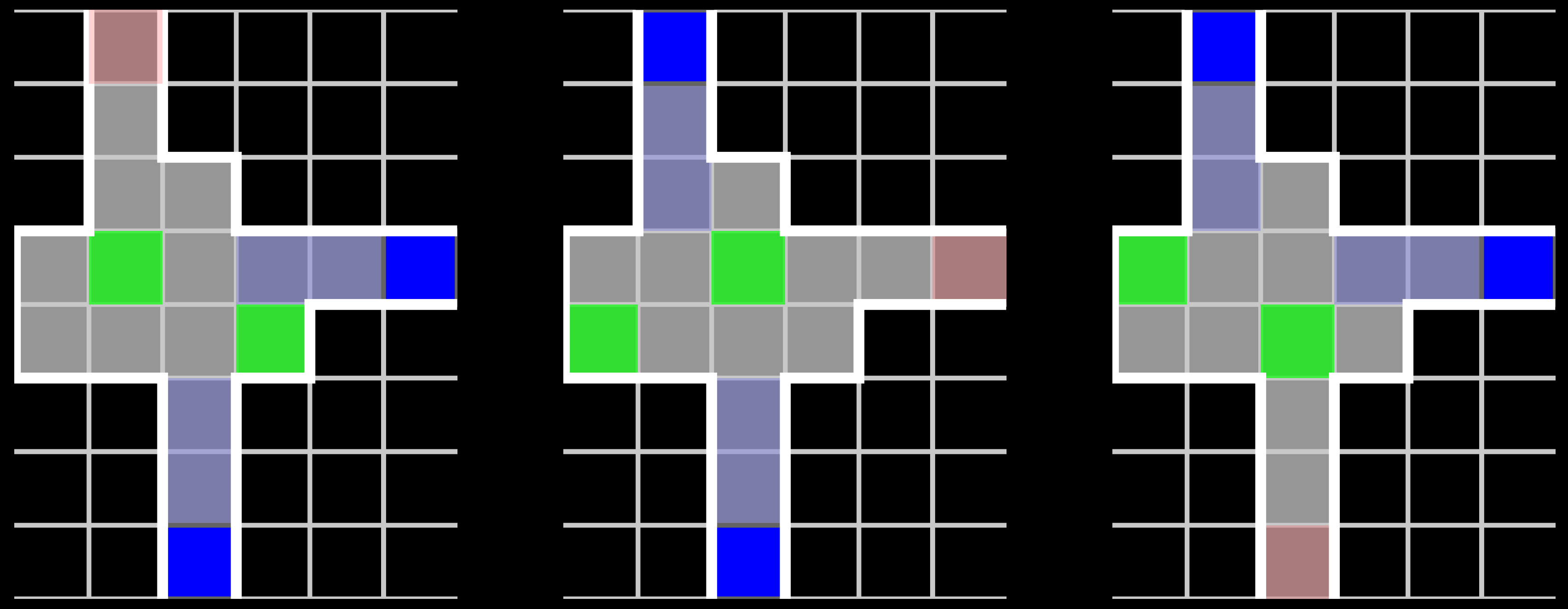
3 guards (necessary and sufficient)

FFT:



2 guards (necessary and sufficient)

FTT:



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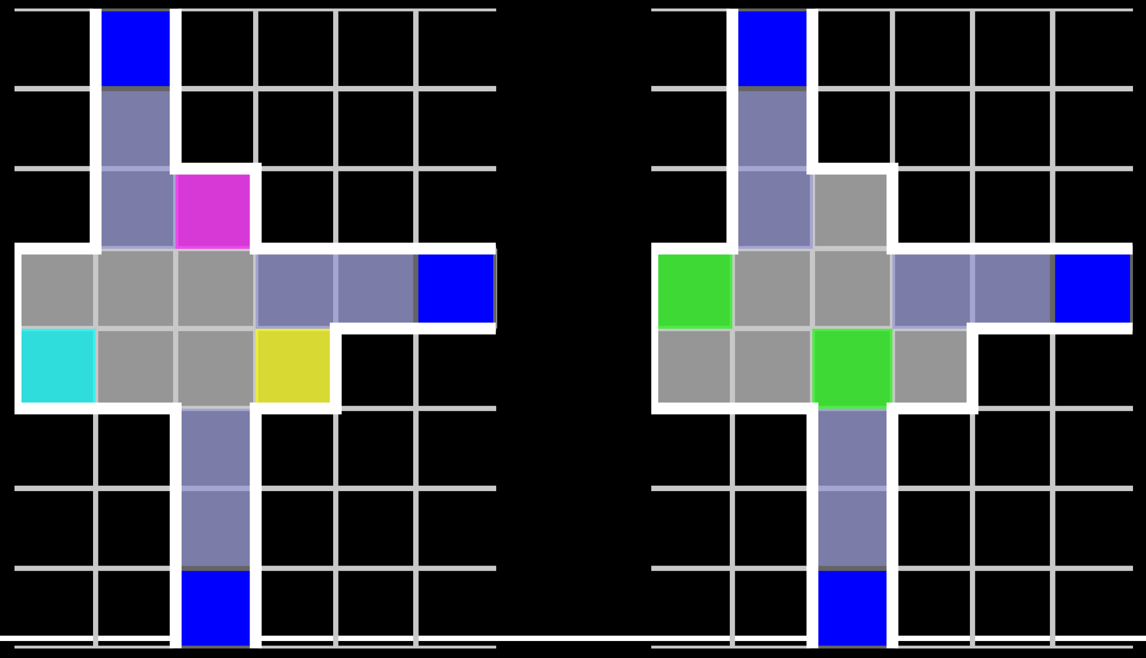
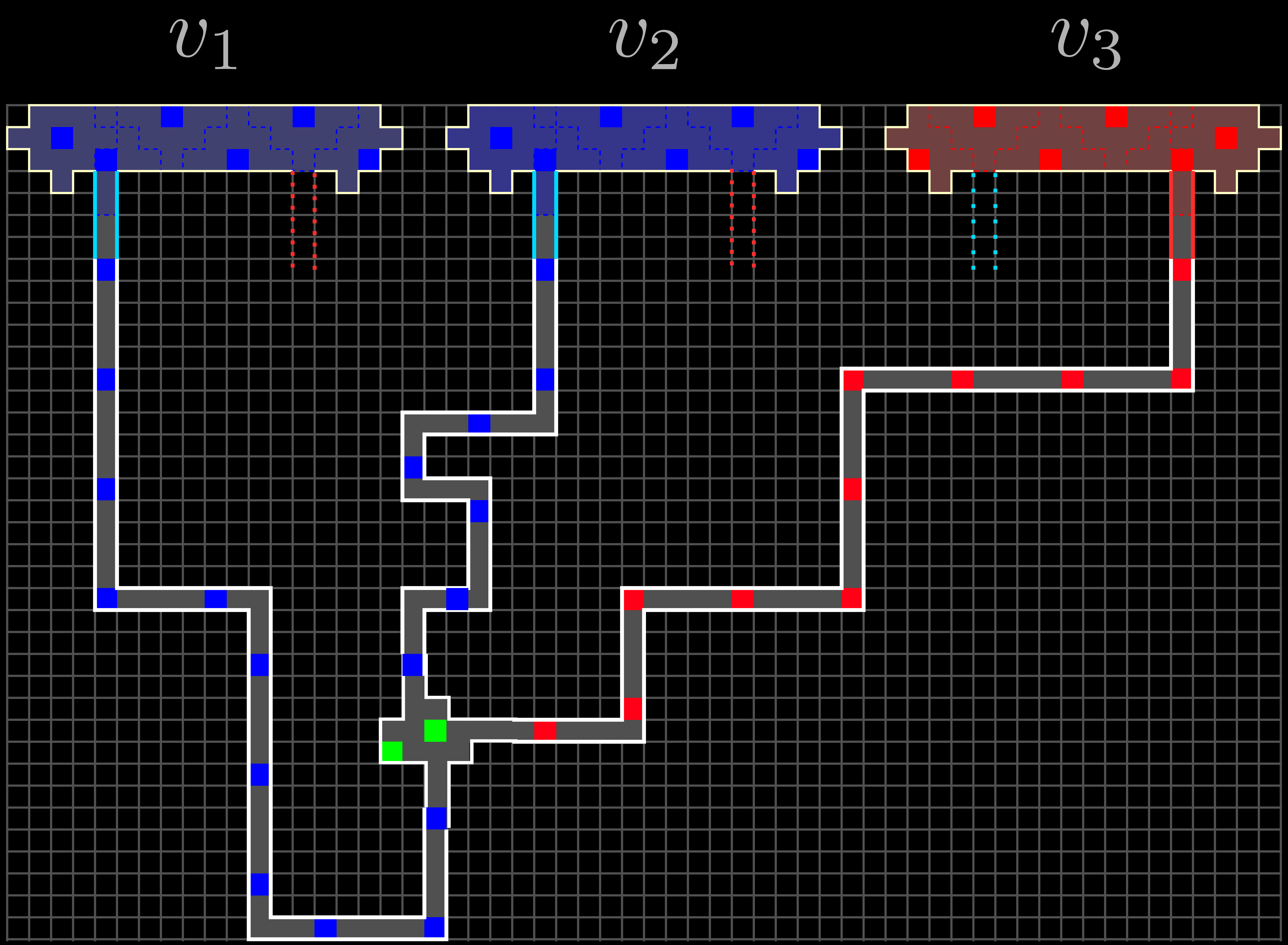


No guard can see all three squares

2 guards (necessary and sufficient)



$$(v_1 \vee v_2 \vee \overline{v_3})$$

Theorem 1: MkGP is NP-complete for k=2 in polyominoes with holes.

Equivalent formulation:

Minimum 2-dominating Set Problem is NP-complete in grid graphs.

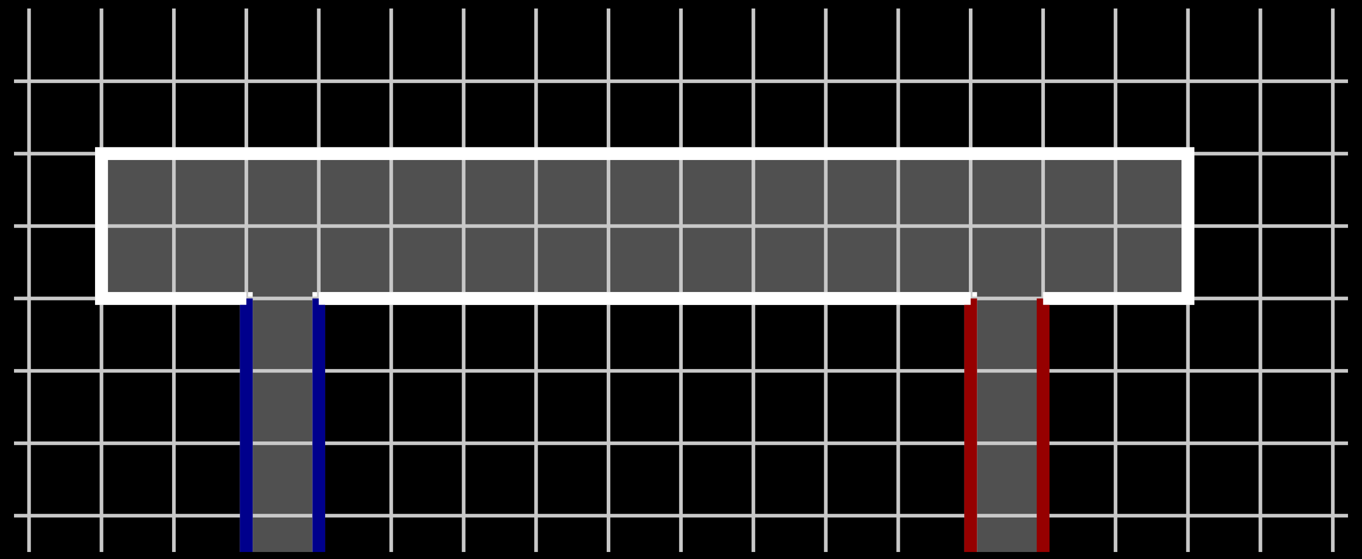
Theorem 2: M*k*GP is NP-complete for *k*=1 in polyominoes with holes.

- Proof by reduction from PLANAR 3 SAT
- Given a set of guards it can be verified in polynomial time whether each unit square of the polyomino is covered

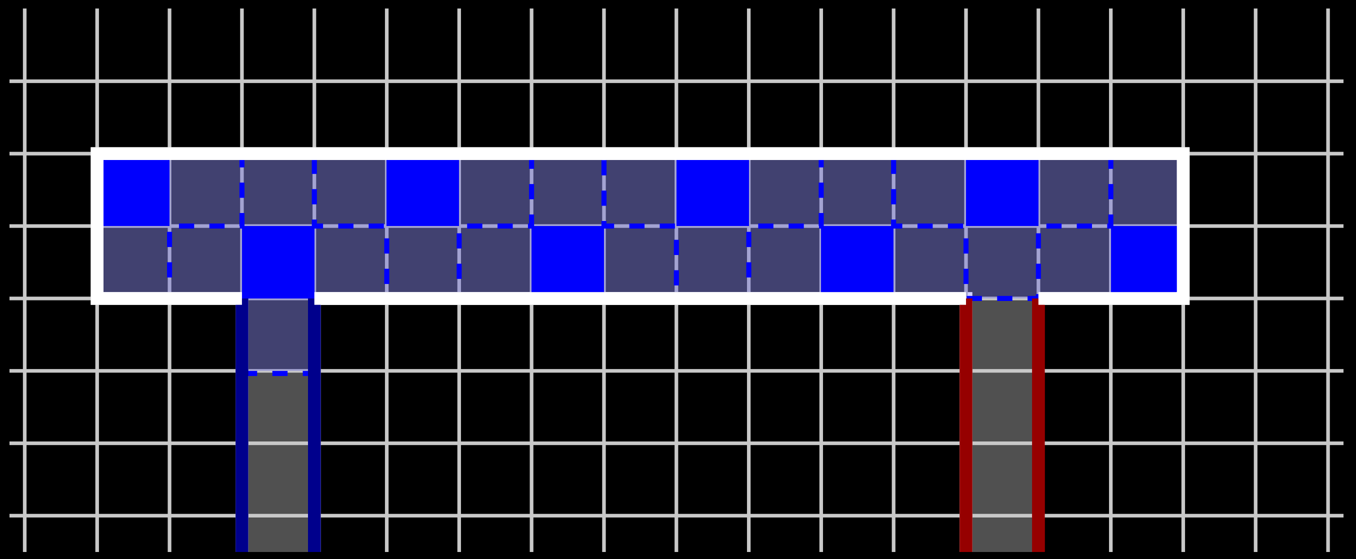
Corridor bending works as before

Variable gadget with two corridor gadgets:

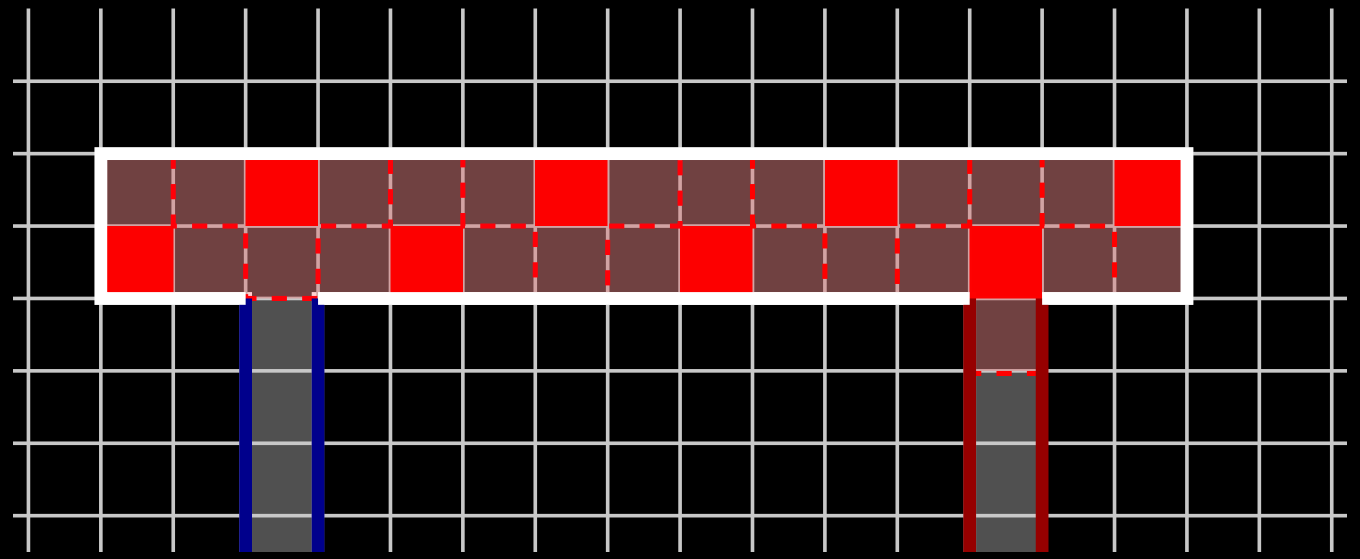
- In case variable appears in clause
- In case negated variable appears in clause



Corridor propagates variable value

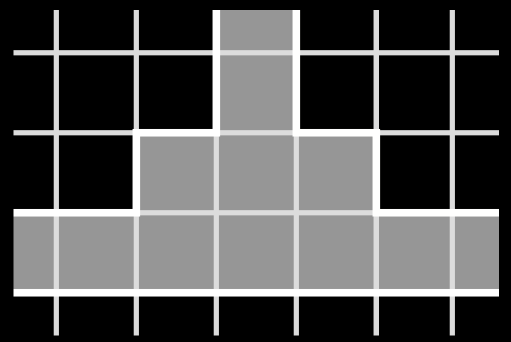


"true"

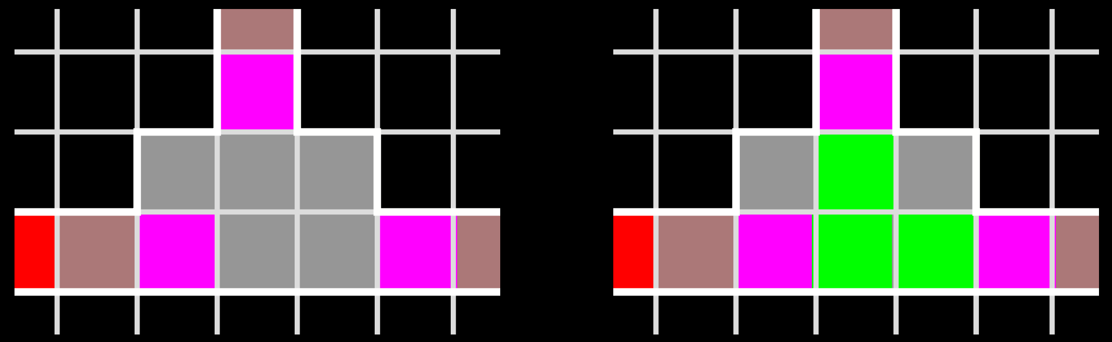


"false"

Clause gadget



FFF: 3 witnesses - pairwise disjoint visibility regions



3 guards (necessary and sufficient)

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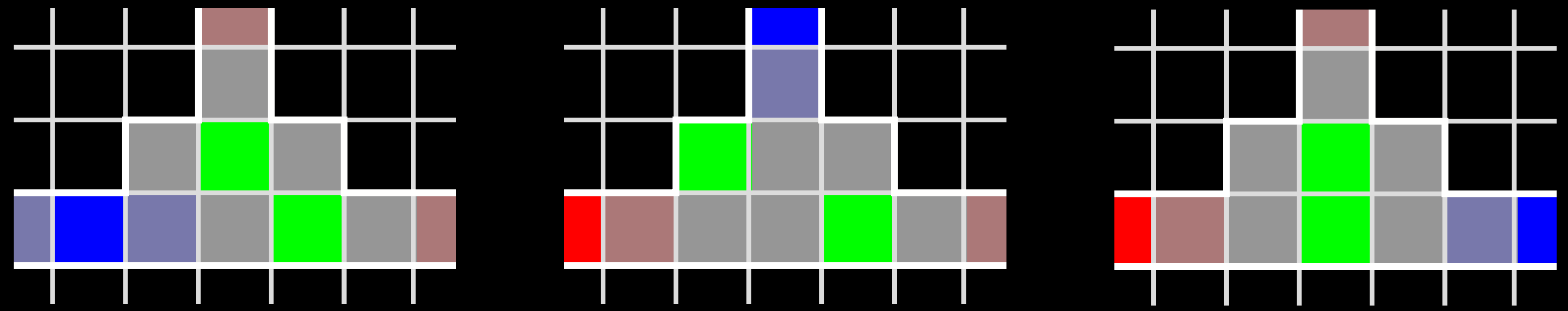

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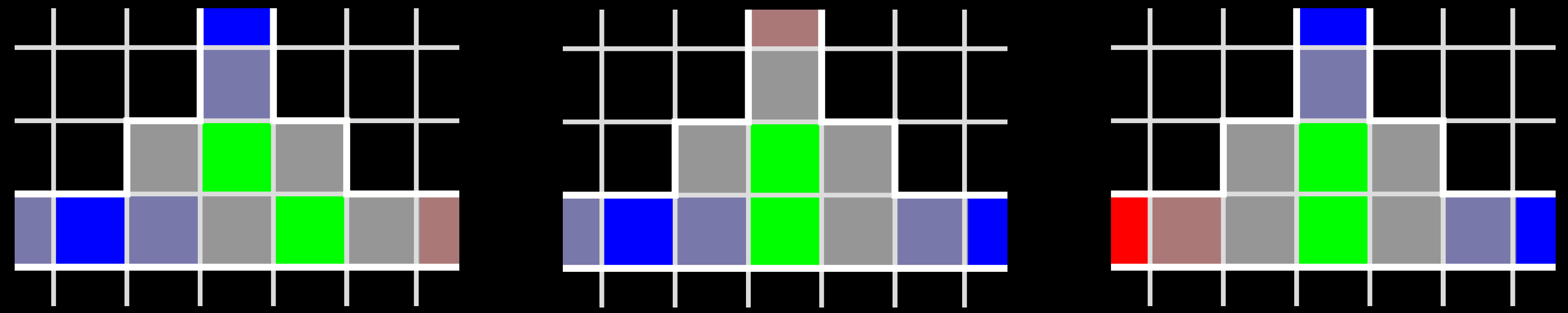

2 guards (necessary and sufficient)

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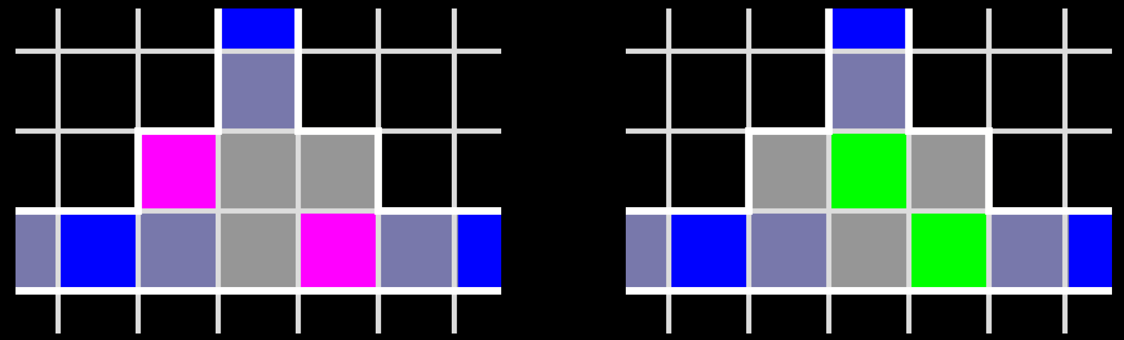

2 guards (necessary and sufficient)

2 witnesses pairwise disjoint visibility regions

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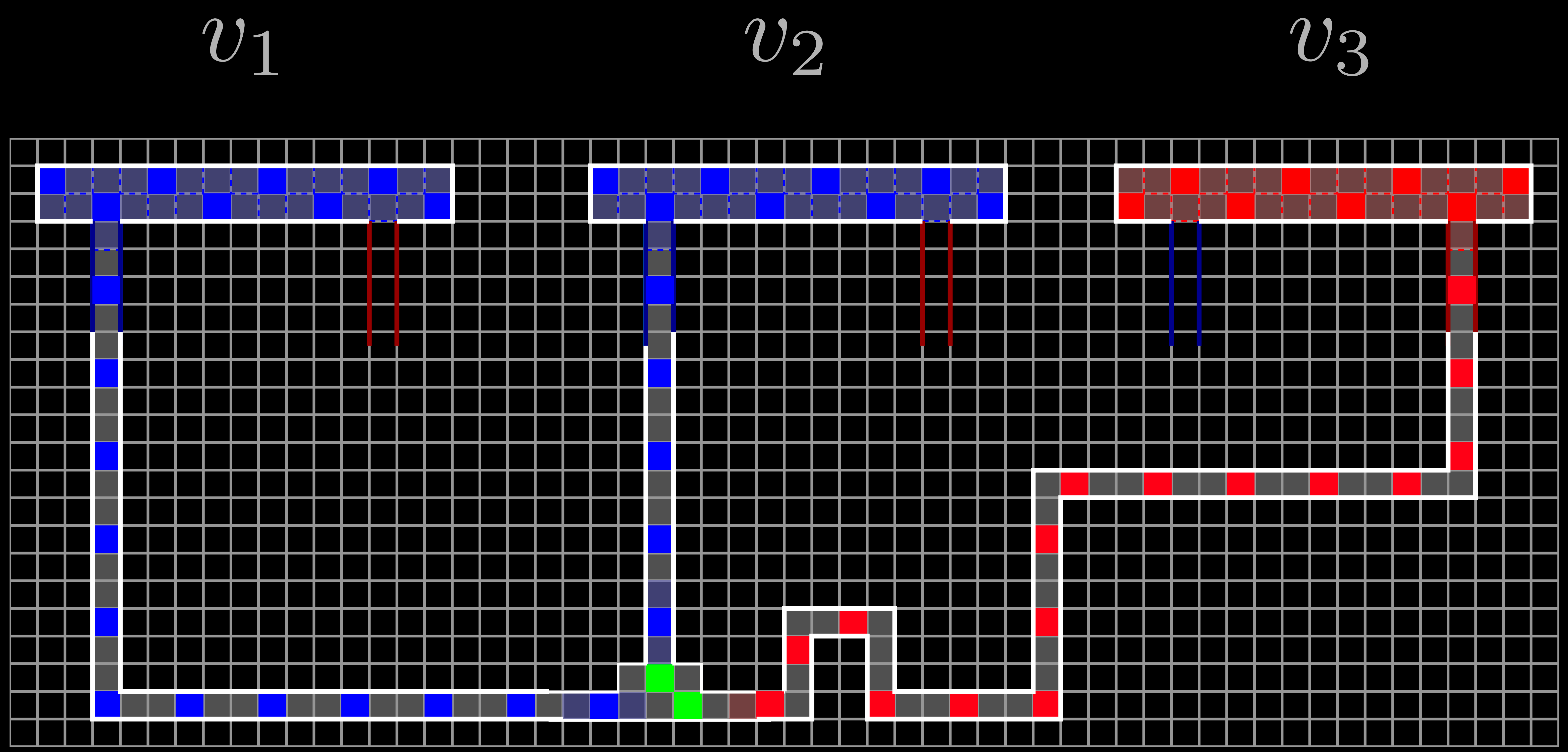


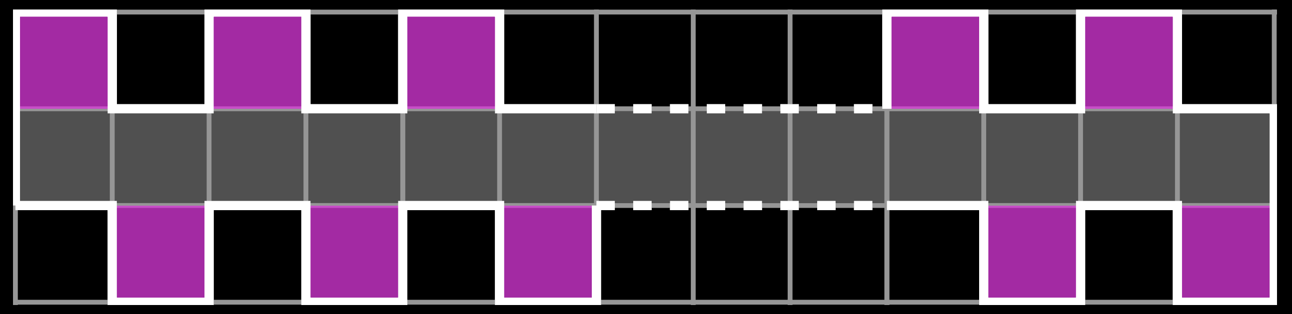
$$(v_1 \lor v_2 \lor \overline{v_3})$$

Art Gallery Theorems

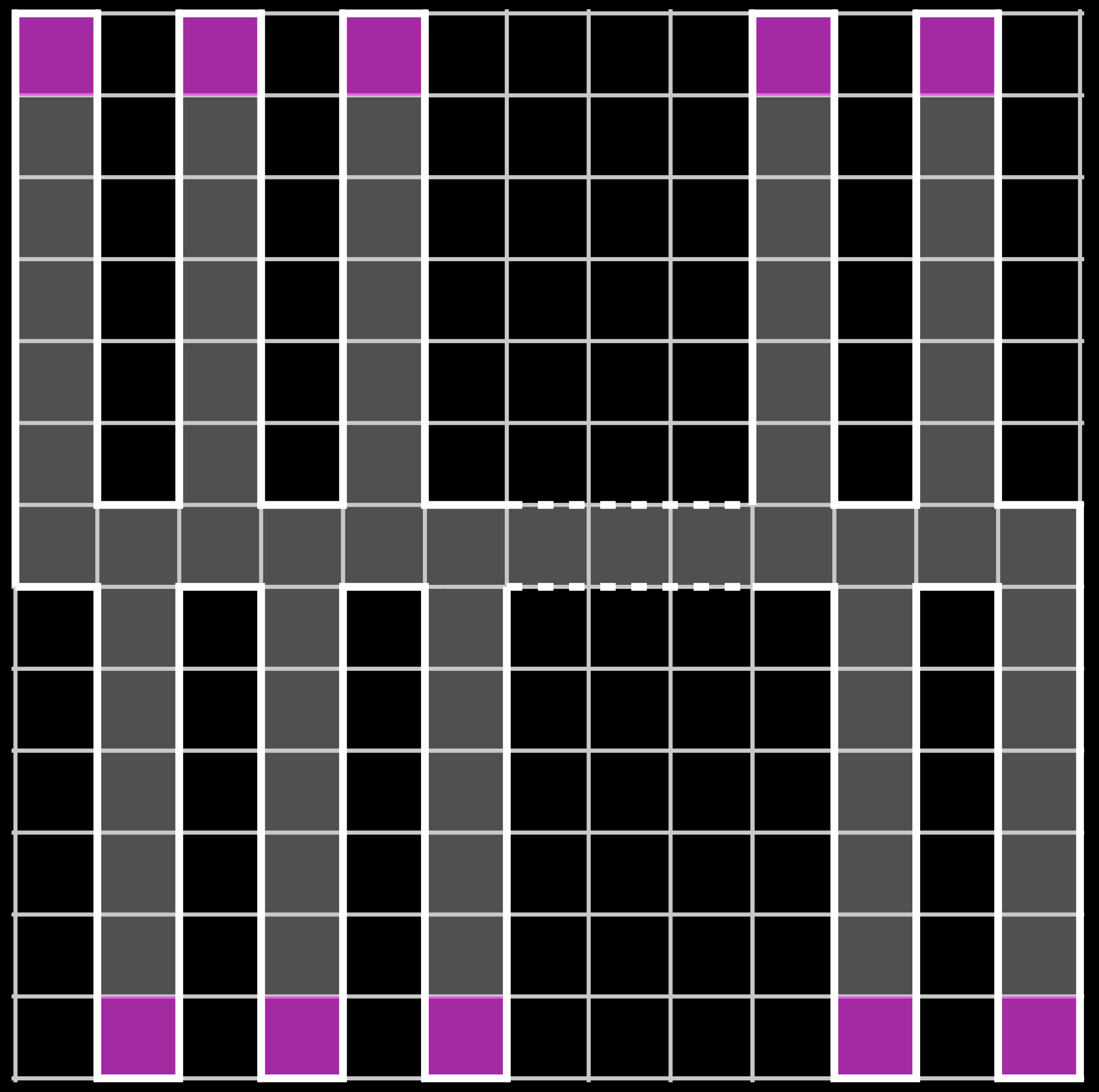
Theorem 3: There exist simple polyominoes with m unit squares that require $\lfloor \frac{m}{k+1} \rfloor$ guards to cover their interior under k-hop visibility.

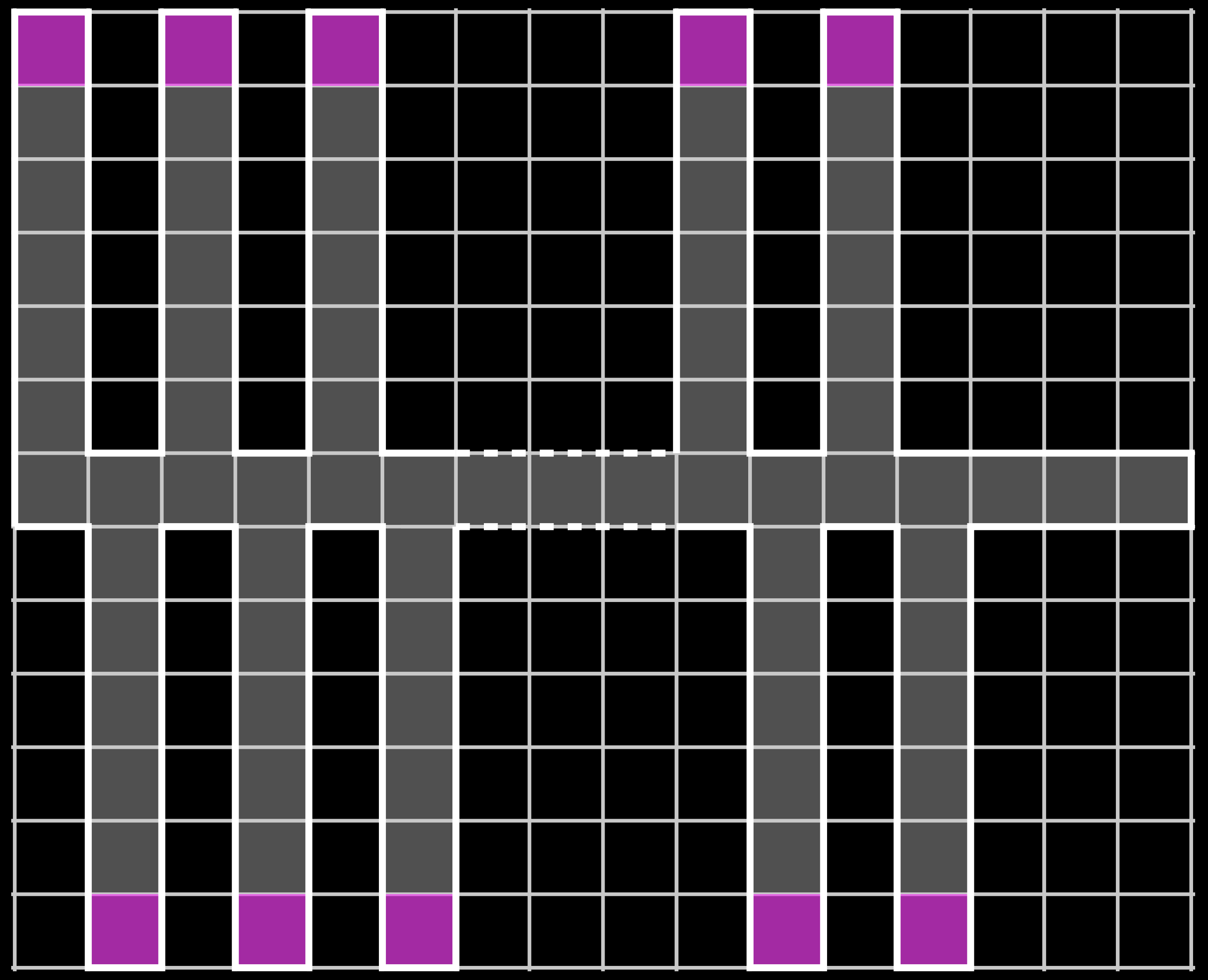
m not divisible by *k+1*
➜ we add *x=(m* mod *(k+1))* unit squares to the right of the shaft

k=1



k=6



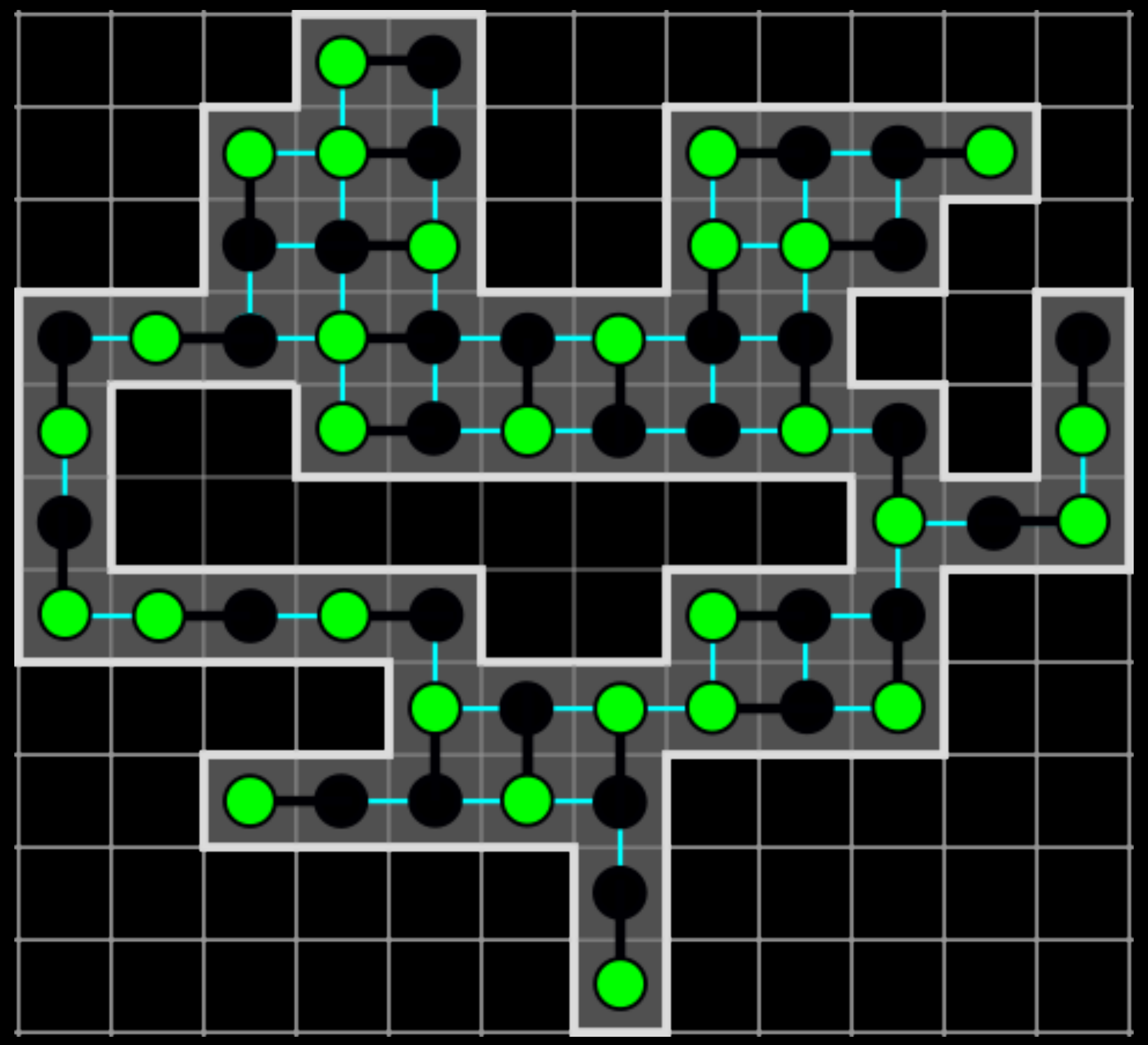


Theorem 4: $\lfloor \frac{m}{k+1} \rfloor$ guards are always sufficient and sometimes necessary to cover a polyomino with m unit squares under k-hop visibility for $k \in \{1,2\}$.

Proof:

k=1:

- Use dual grid graph G_P
- Compute a maximum matching M in the dual grid graph G_P
- Every vertex in G_P that is not matched is adjacent to matched vertices only
- For each matching edge {v,w} unmatched vertices only adjacent to v or w
- For each matching edge: place guard at unit square adjacent to unmatched vertices (if any)

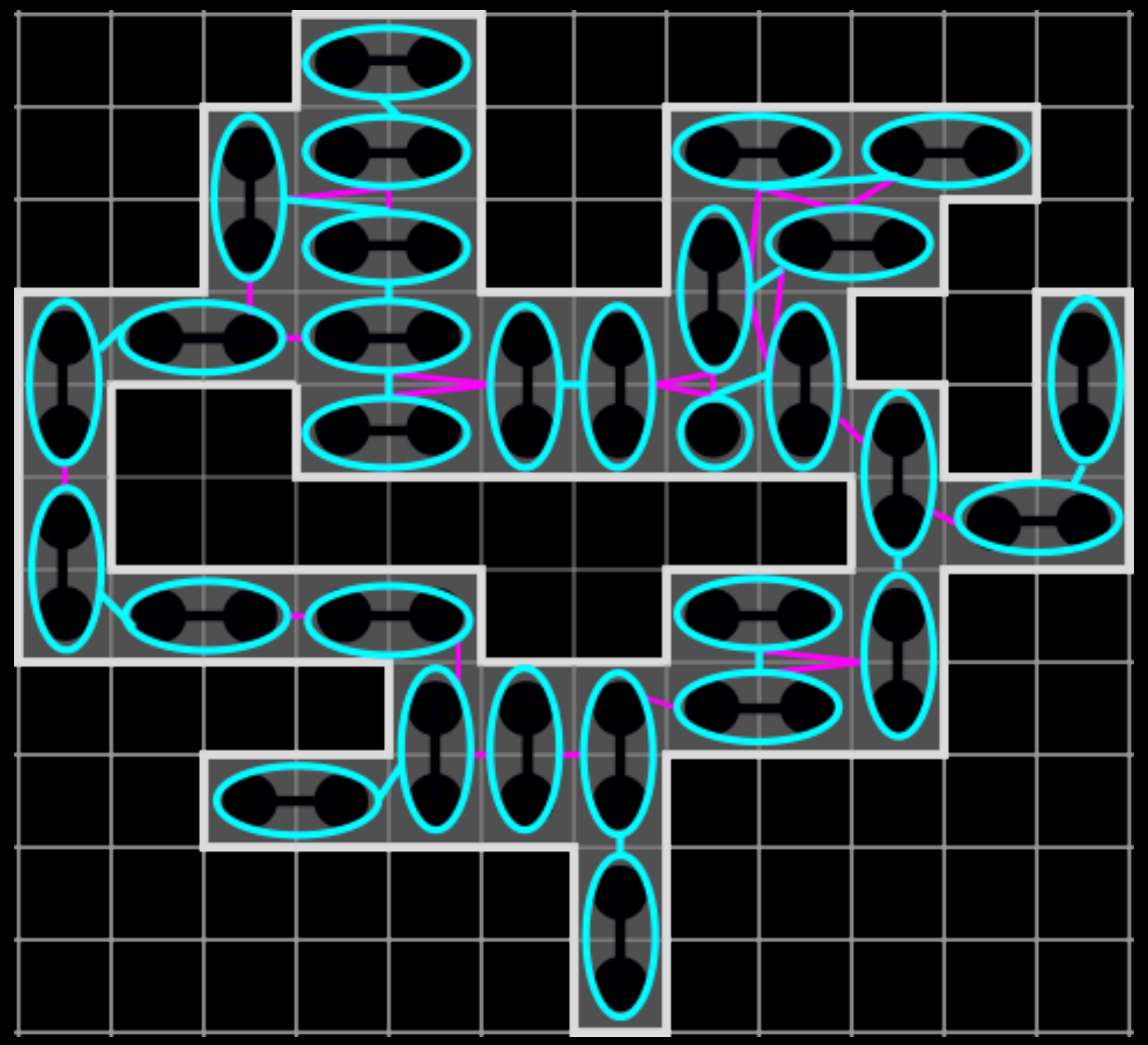


Theorem 4: $\lfloor \frac{m}{k+1} \rfloor$ guards are always sufficient and sometimes necessary to cover a polyomino with m unit squares under k-hop visibility for $k \in \{1,2\}$.

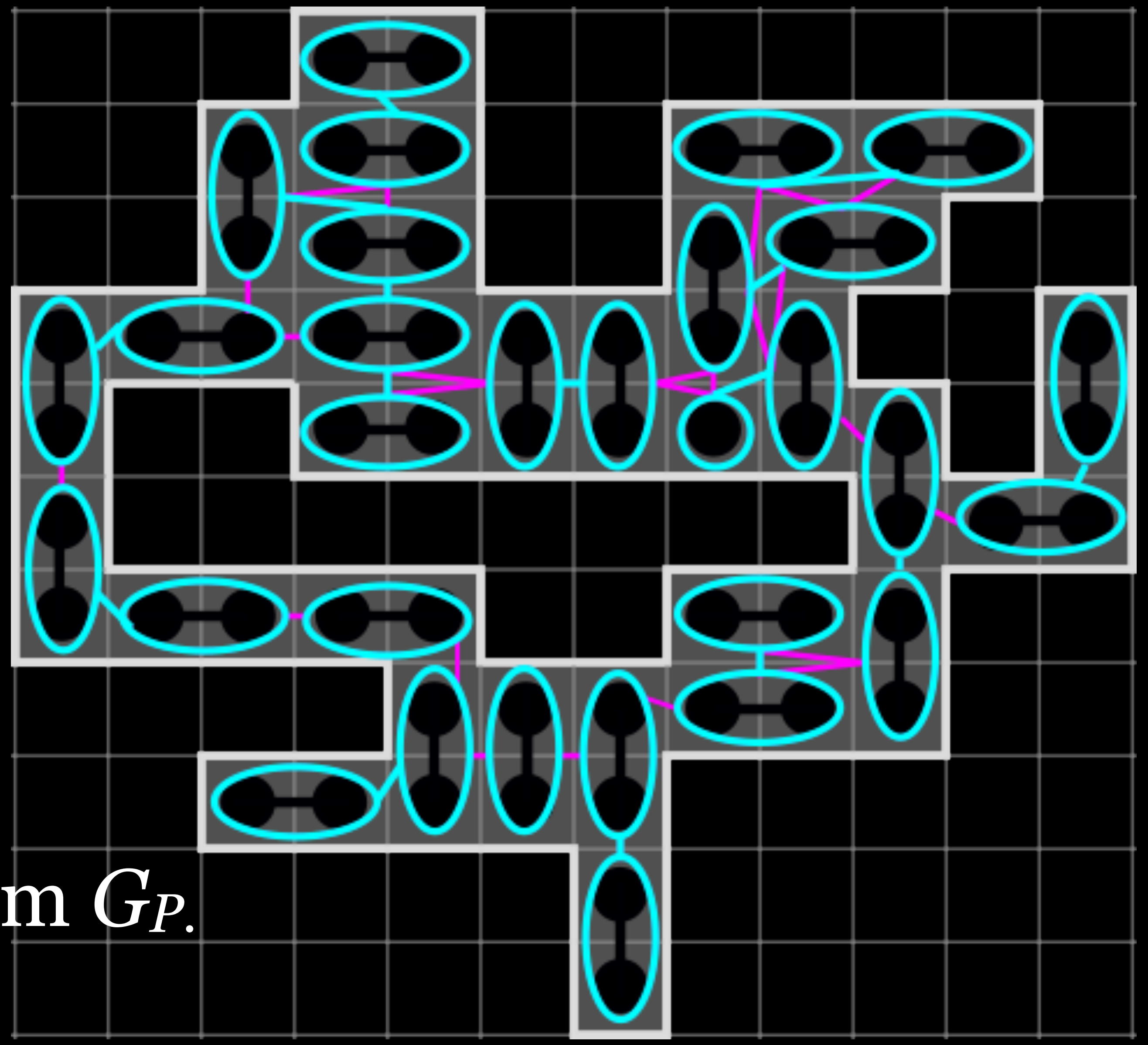
Proof:

k=2:

- Again: Compute a maximum matching M in the dual grid graph G_P
- Graph G_M based on M:
  - Vertex for each matching edge
  - Vertex for each unmatched vertex
  - $\{v,v'\} \in V(G_M)$ if:
    - For $v=v_{\{u,u'\}}$, $v'=v'_{\{w,w'\}}$: if at least one of the edges $\{u,w\}$, $\{u,w'\}$, $\{u',w\}$ or $\{u',w'\}$ $\in E(G_P)$
    - For $v=v_{\{u,u'\}}$, $v'=v'_w$: if at least one of $\{u,w\}$ or $\{u,w'\}$ $\in E(G_P)$
- Compute maximum matching M' in G_M

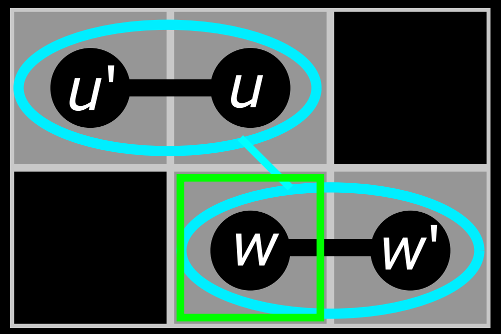


Theorem 4: guards are always sufficient and sometimes necessary to cover a polyomino with m unit squares under k-hop visibility for $k\in\{1,2\}$.



Proof:

k=2: Case distinction

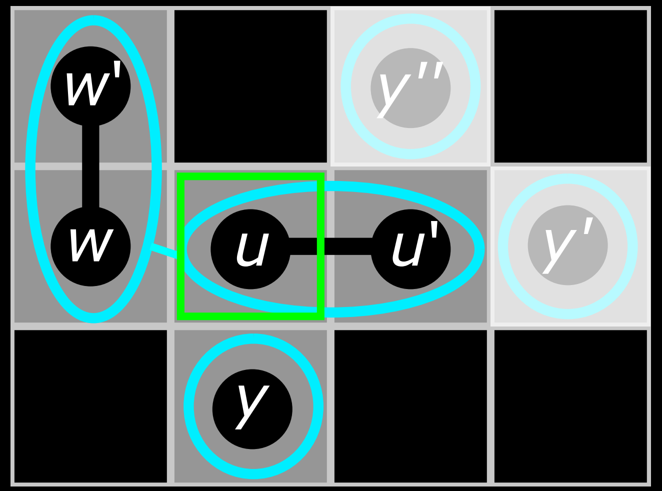


e=$\{v_{\{u,u'\}}, v_{\{w,w'\}}\}\in M'$ not adjacent to any other unmatched vertex

Single guard covers 4 unit squares

e=$\{v_{\{u,u'\}}, v_{\{w,w'\}}\}\in M'$ adjacent to unmatched vertices, all of which represent one vertex from G_P.

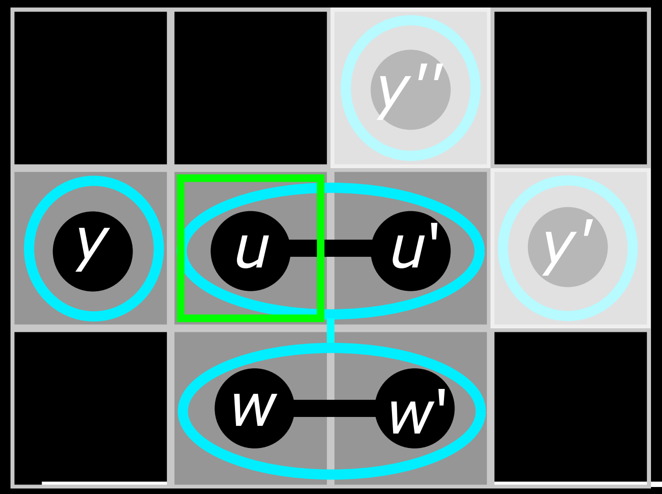
W.l.o.g. all unmatched vertices adjacent to $v_{\{u,u'\}}$



$\{u,w\}\in E(G_P)$ or $\{u,w'\}\in E(G_P)$, but $\{u',w\}\notin E(G_P)$ and $\{u',w'\}\notin E(G_P)$

Single guard on u covers u, u', w, w' and all unmatched vertices adjacent to covered $v_{\{u,u'\}}$

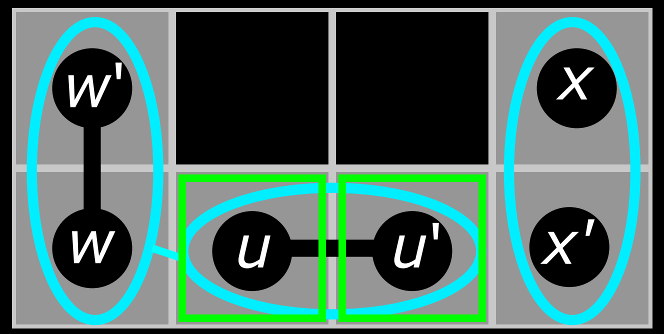
Single guard covers at least 5 unit squares



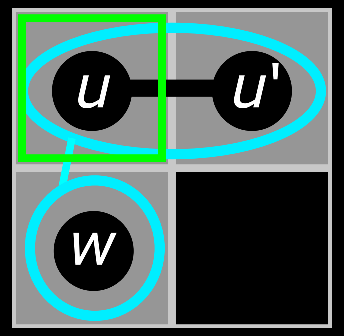
$\{u,w\}\in E(G_P)$ and $\{u',w'\}\in E(G_P)$ (or $\{u',w\}\in E(G_P)$ and $\{u',w'\}\in E(G_P)$)

Single guard on u (or u') covers u, u', w, w' and all unmatched vertices adjacent to covered $v_{\{u,u'\}}$

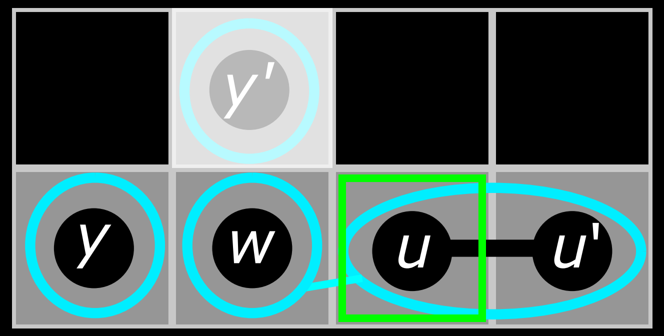
Single guard covers at least 5 unit squares



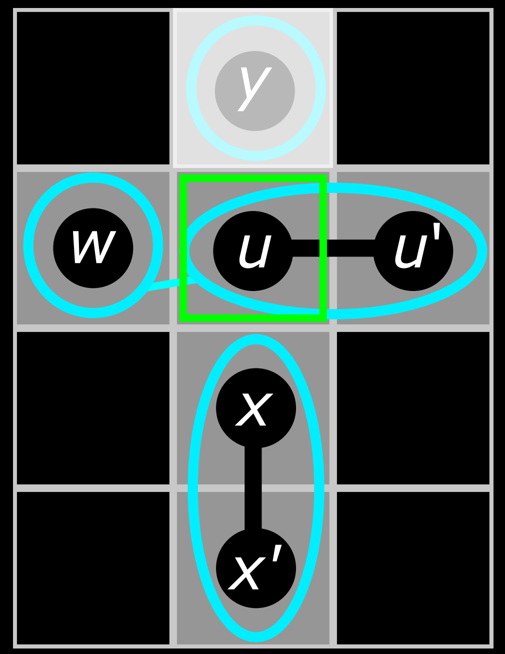
Two guards cover at least 6 unit squares



Single guard covers 3 unit squares

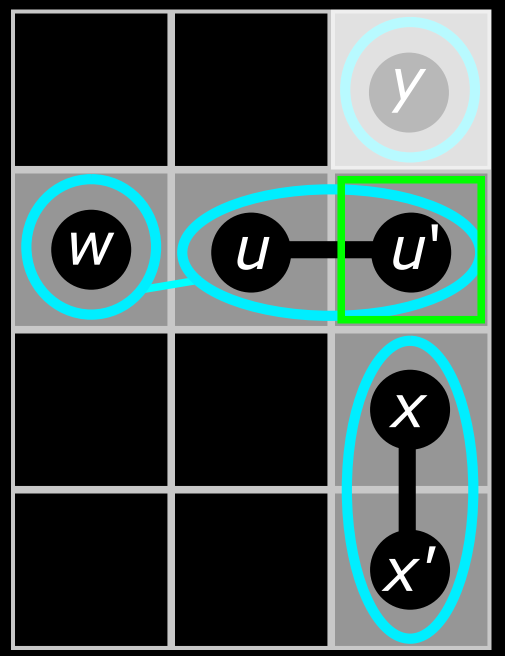


Single guard covers at least 4 unit squares

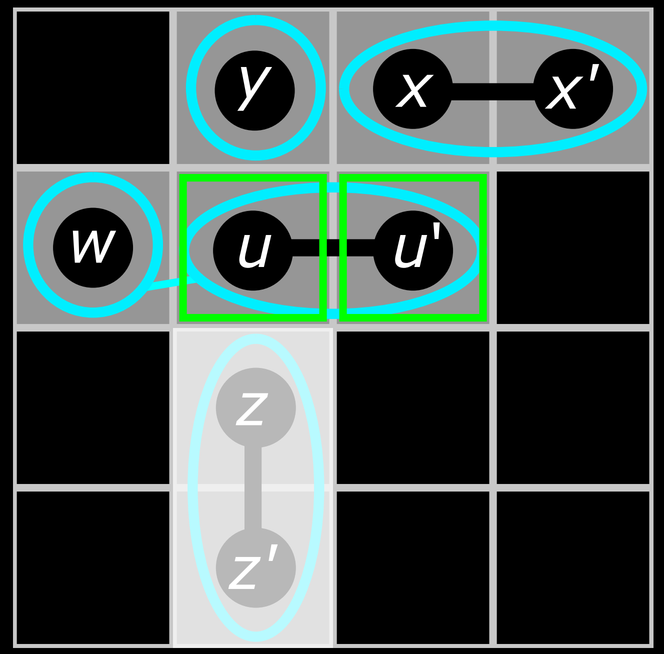


Single guard covers at least 5 unit squares

→ Each guard covers at least 3 unit squares
→ $\lfloor m/(k+1) \rfloor = \lfloor m/3 \rfloor$ guards always sufficient



Single guard covers at least 5 unit squares



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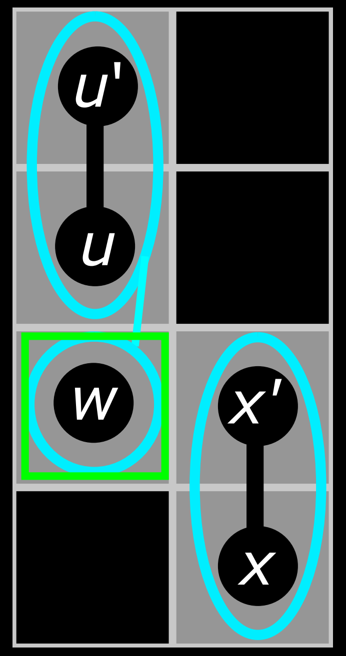
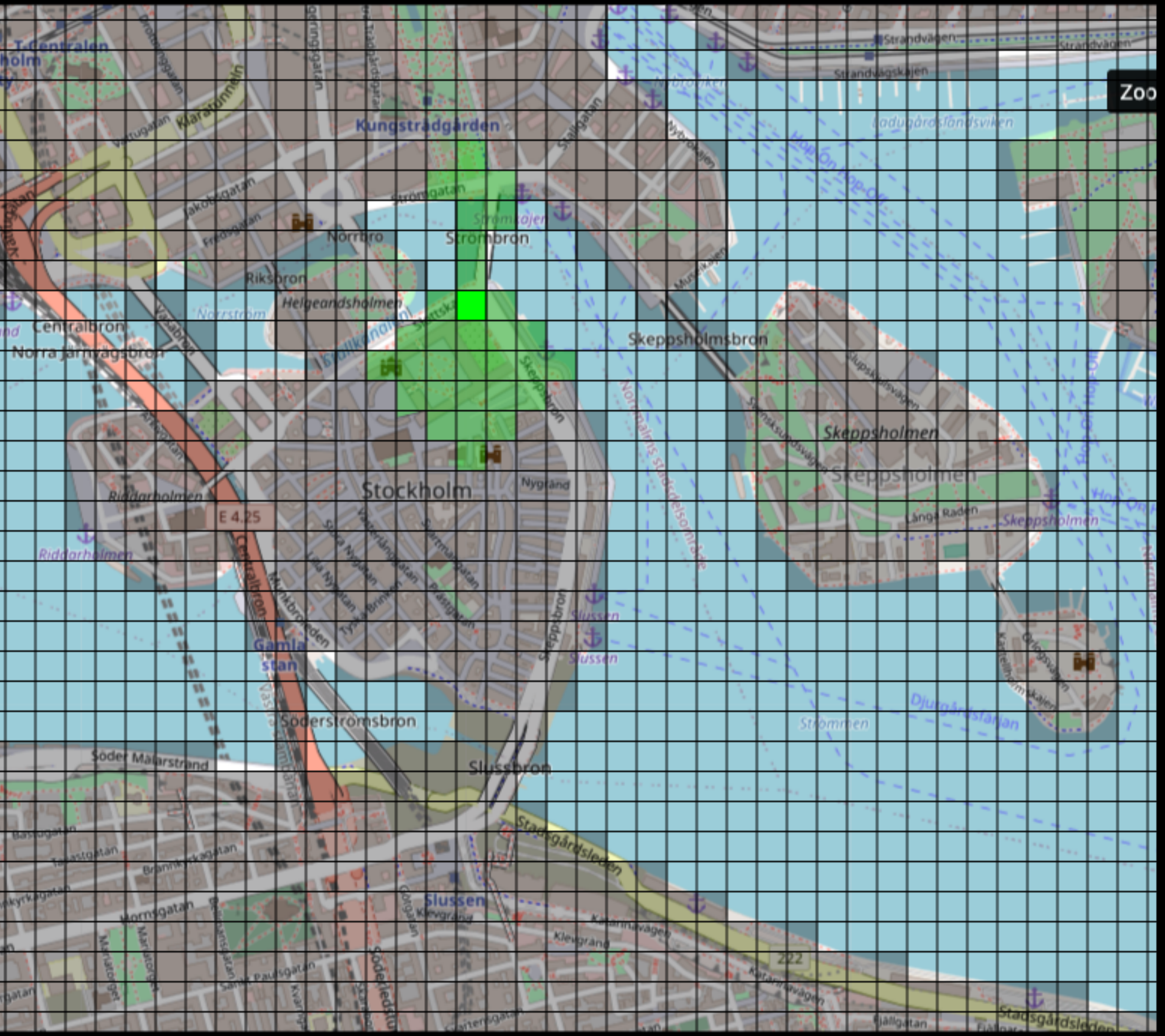


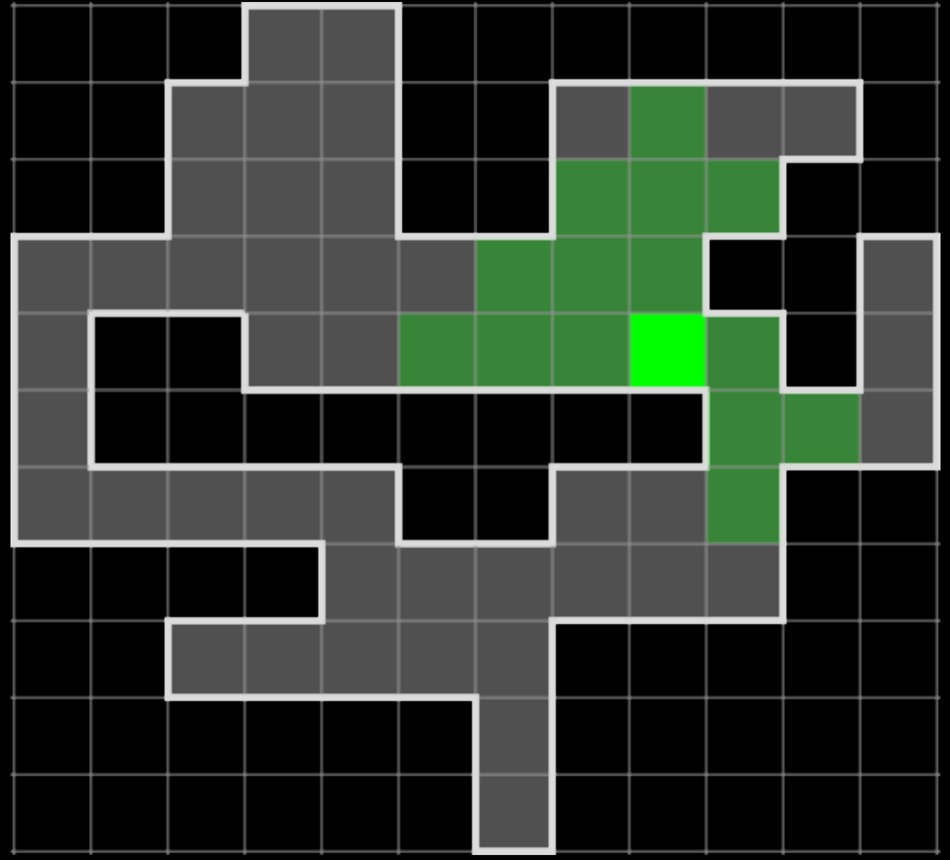
Single guard covers at least 5 unit squares

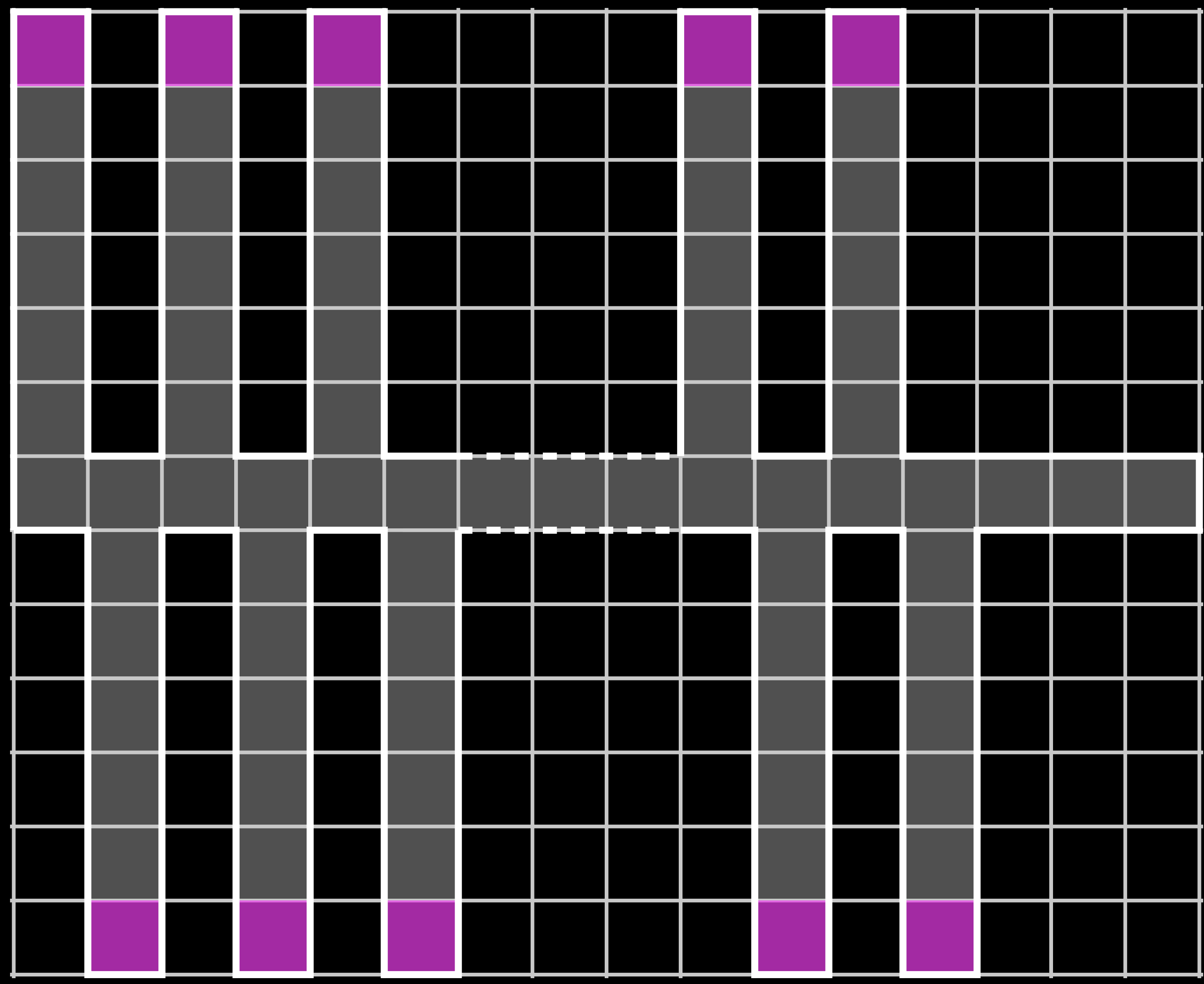
Open Problems

Open Problems

- Computational complexity in simple polyominoes (grid graphs without holes)
- Upper bounds on the number of guards for $k \geq 3$
- Approximation algorithms ($k=1$: trivial 5/2-approximation)







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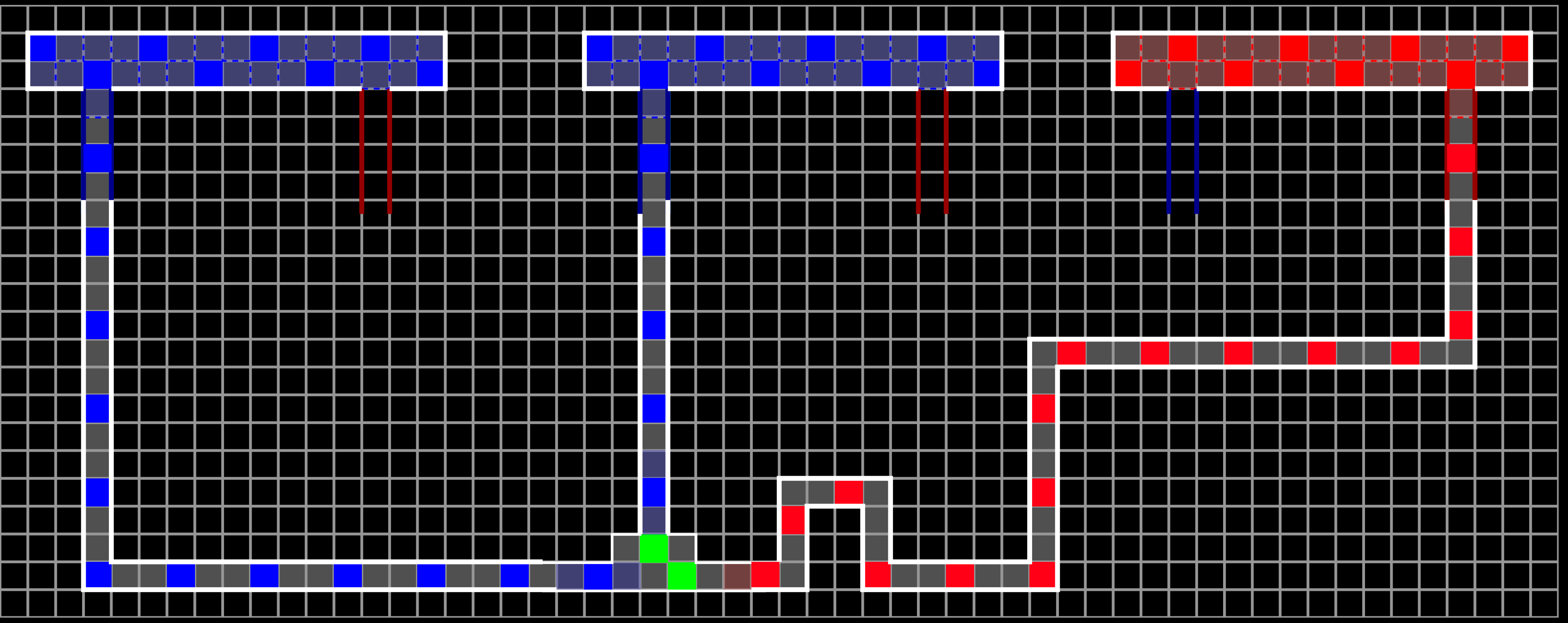

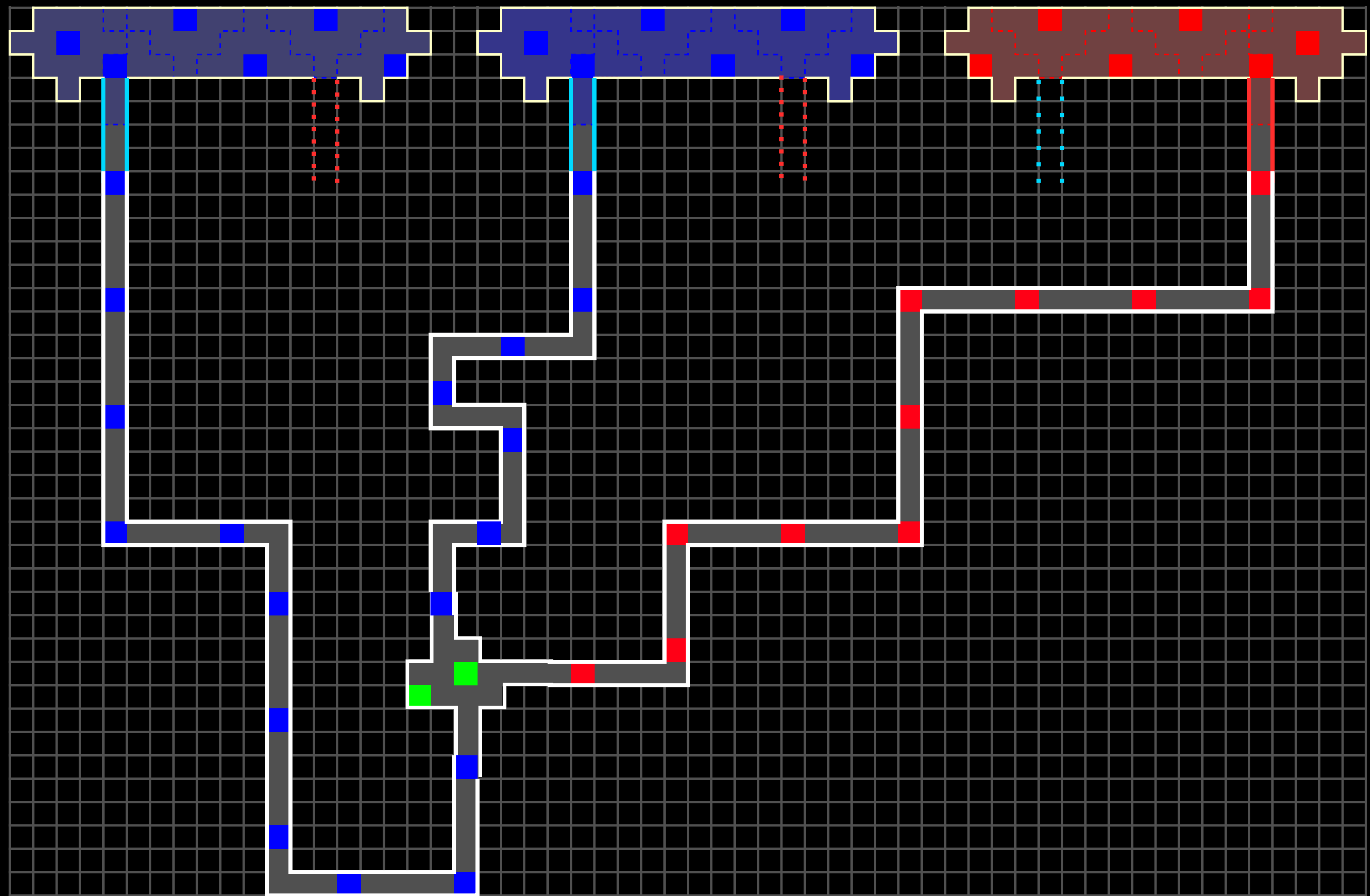

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