Rectangular Spiral Galaxies are Still Hard

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Massachusetts Institute of Technology

Maarten Löffler

Christiane Schmidt

LINKÖPING UNIVERSITY

Polygon Partitions

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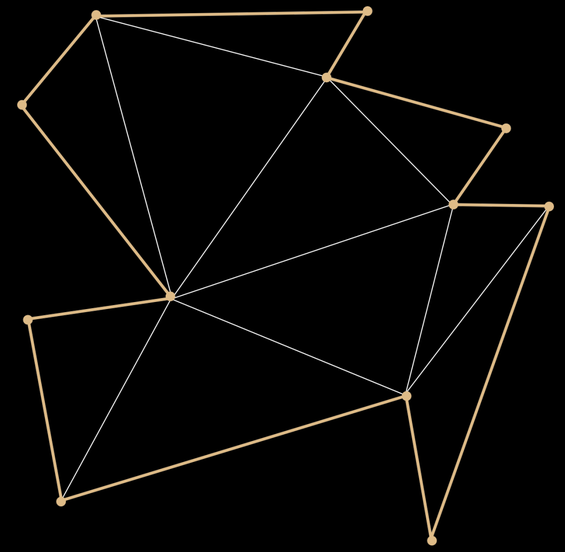
- ★ Triangulation



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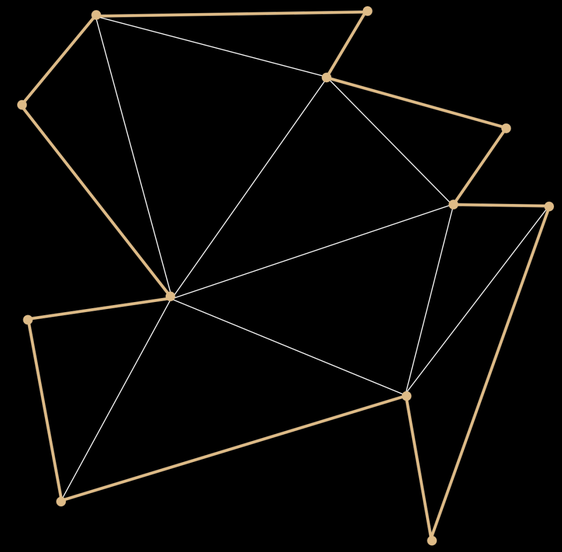

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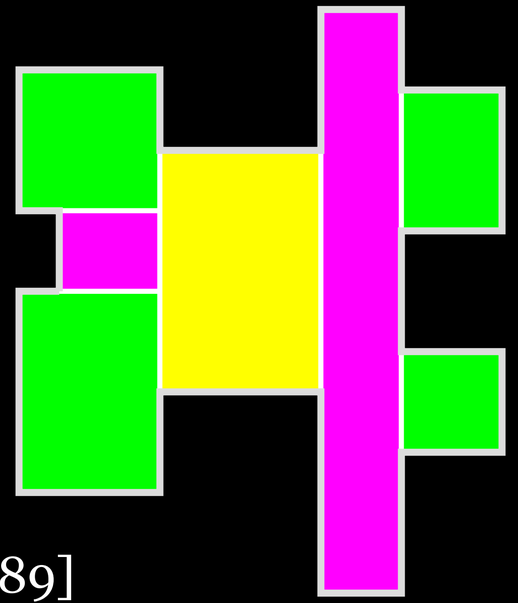

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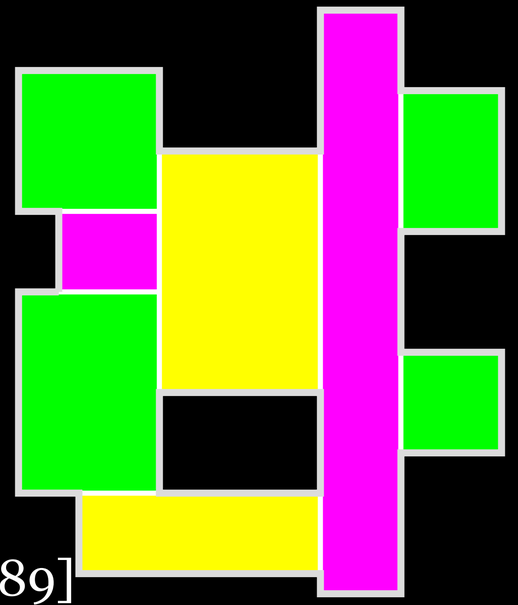

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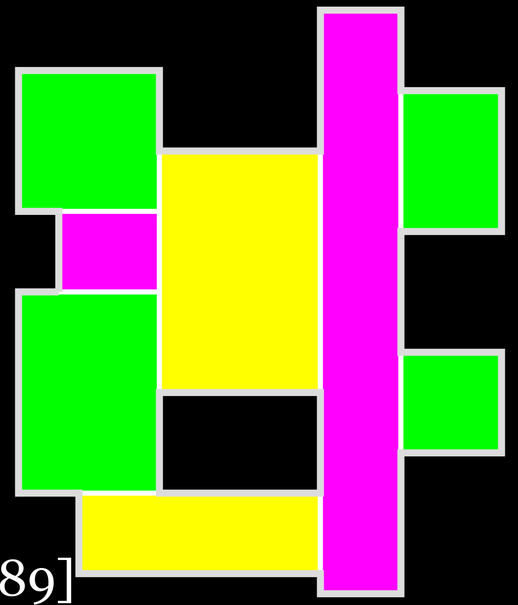

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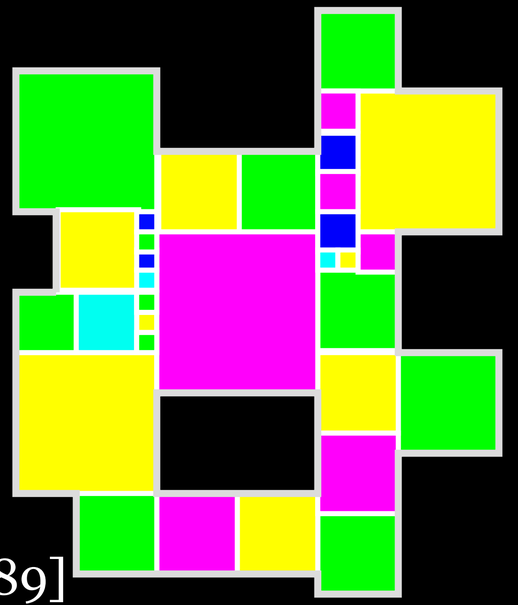

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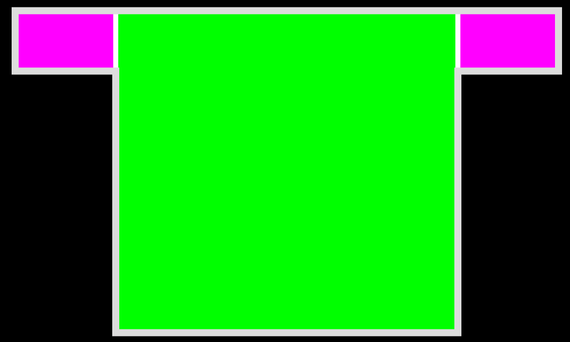

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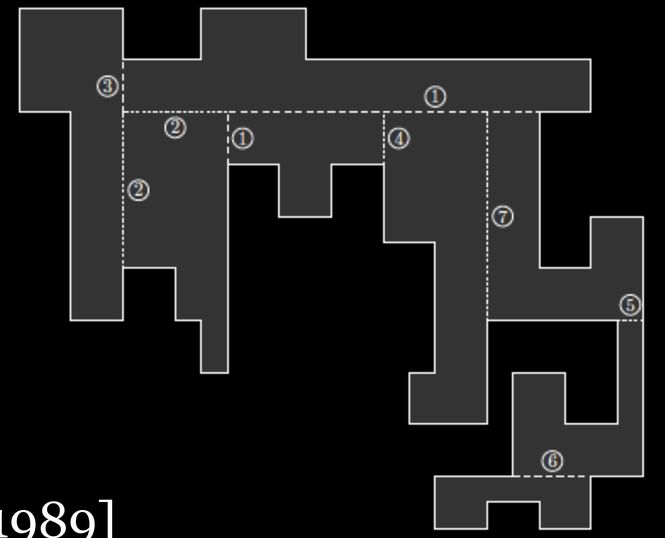
Figure from: Partitioning orthogonal polygons into 8-vertex pieces, with application to an art gallery theorem. Ervin Győri, Tamas Robert Mezei



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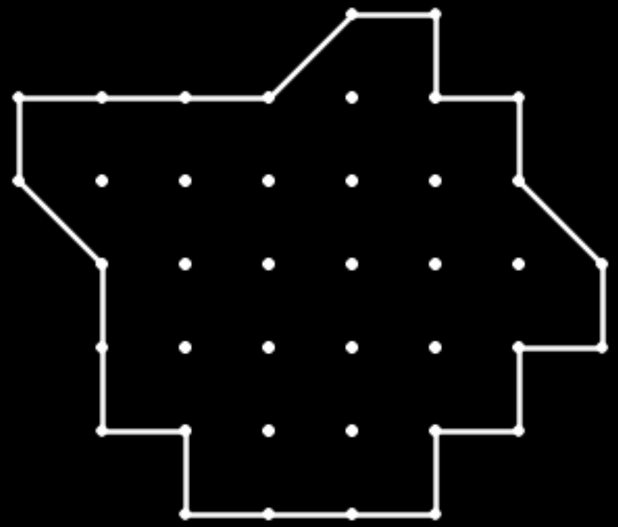
- ★ Partition rectangular polygons into polygons with at most 8 vertices: $\lfloor {}^{3n+4}/_{16} \rfloor$ polygons with O(n) algorithm [Győri, Mezei, 2016]

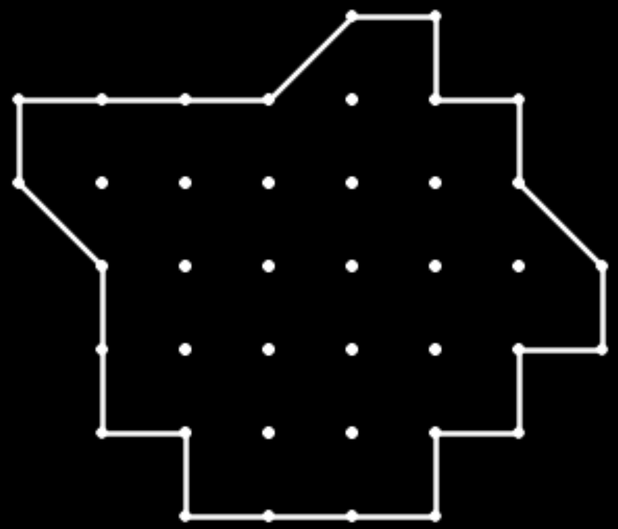
Polygon Partitions



http://puzzlepicnic.com/puzzle?3967

Polygon Partitions

★ Partition into “similar” polygons:



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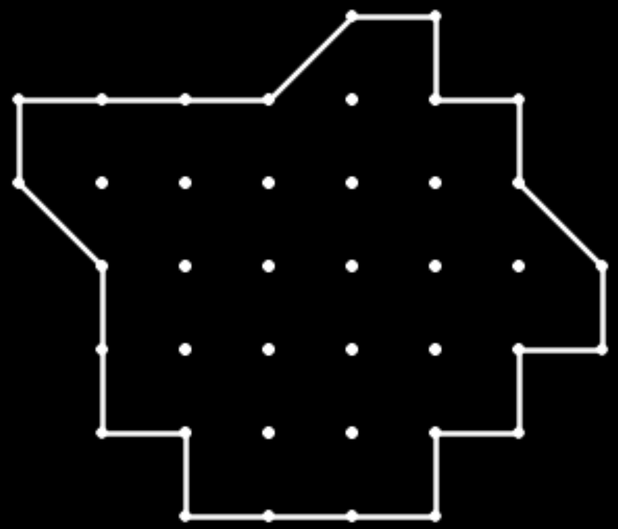

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Several versions, most common: homothetic—same size, can mirror

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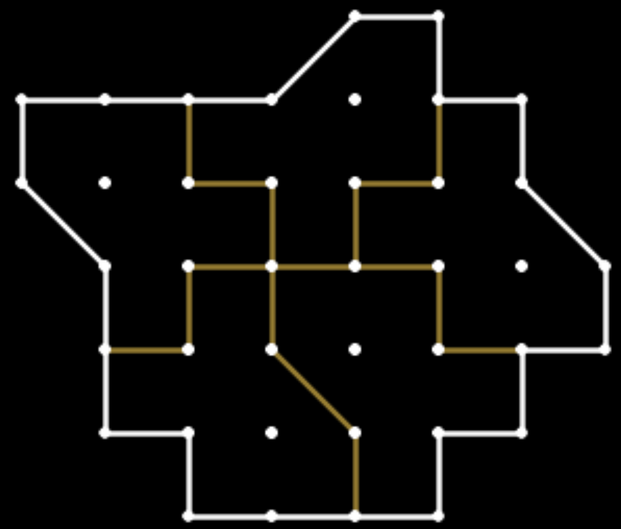

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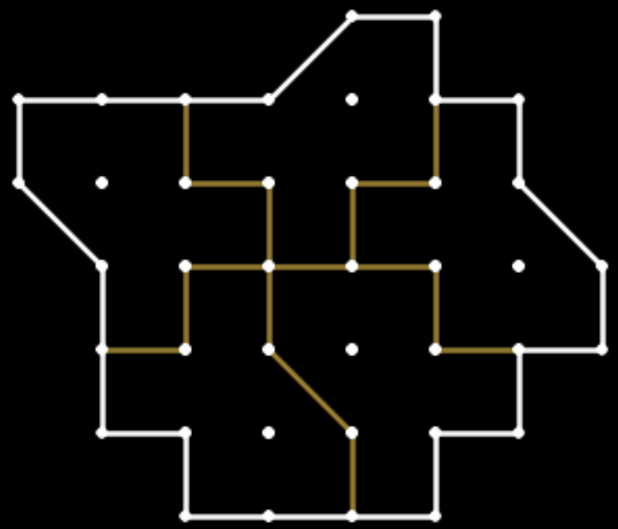

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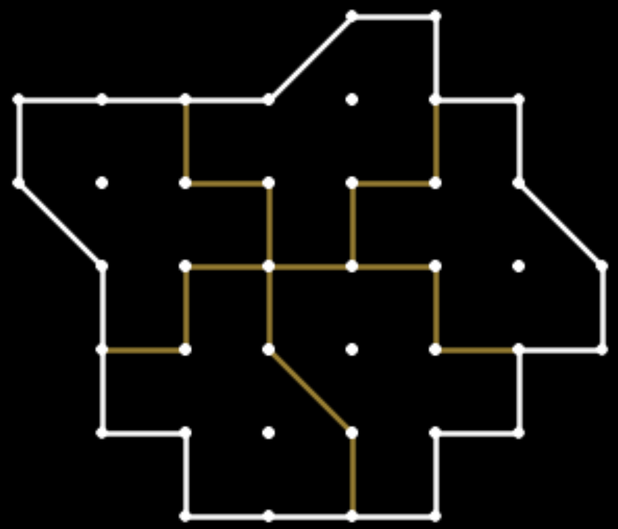

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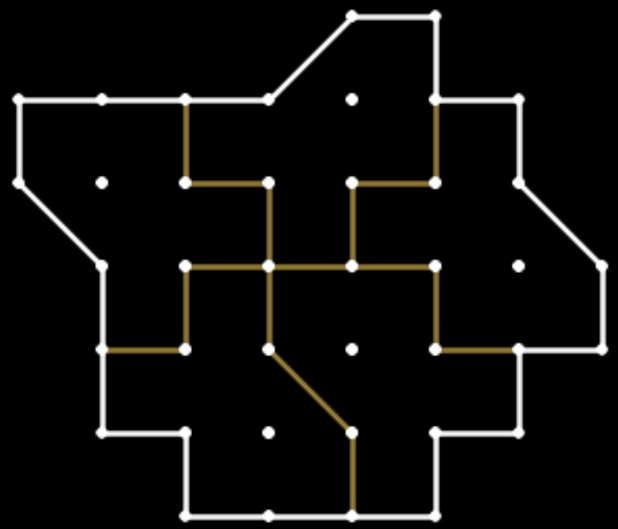

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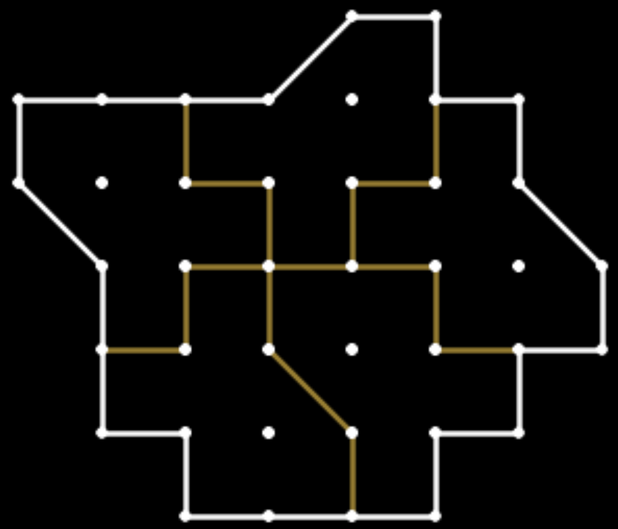
- ★ Rectangular polygons ➜ let’s look at polyominoes



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Spiral Galaxies

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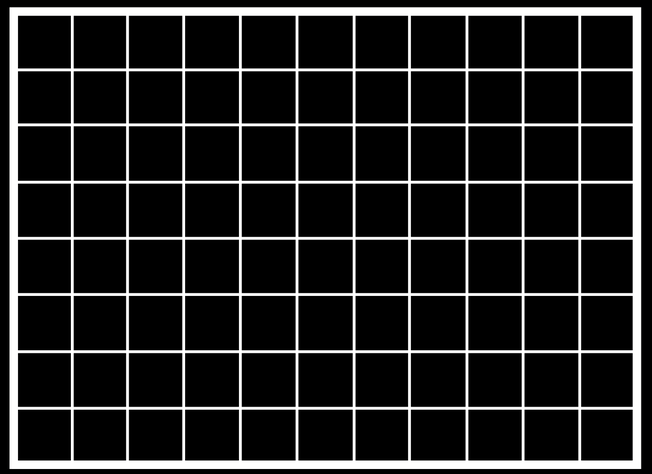

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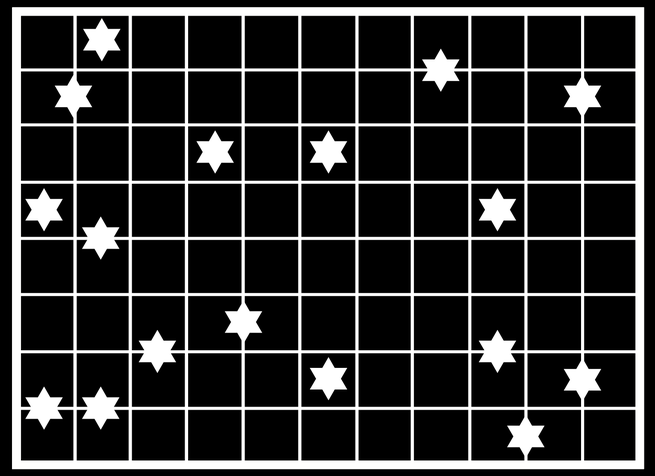

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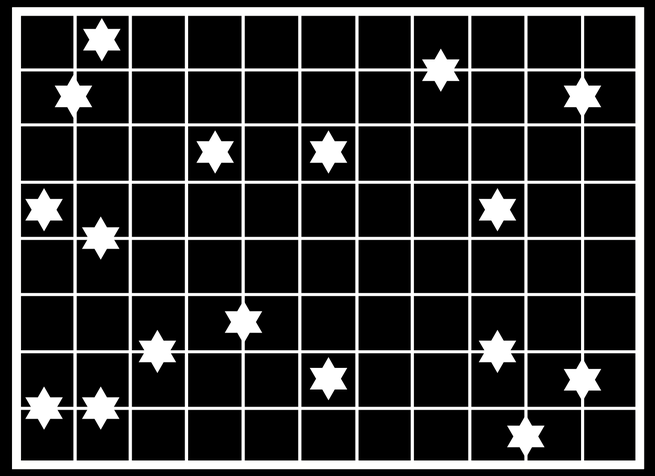

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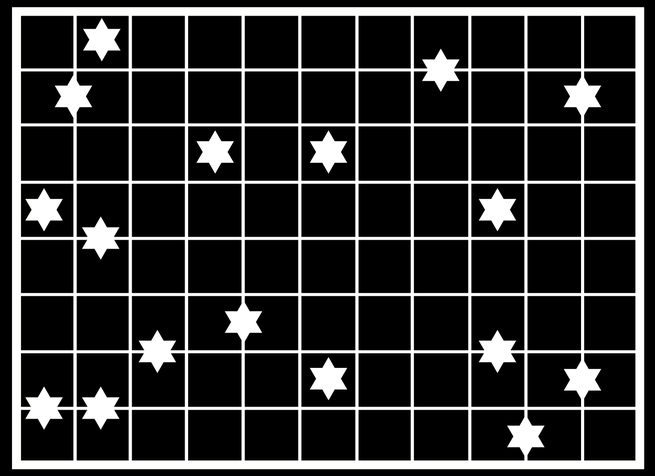

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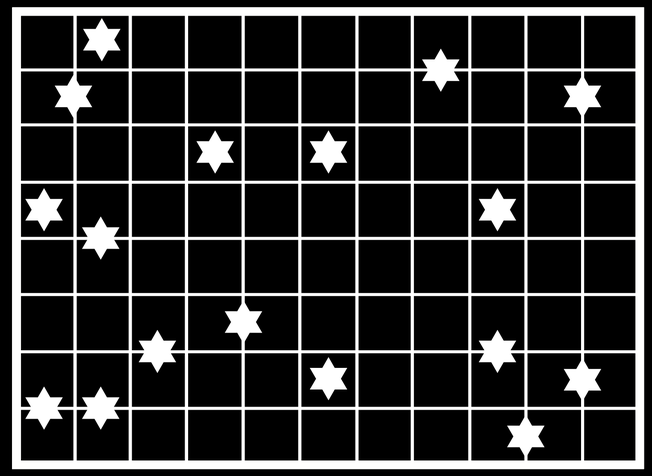

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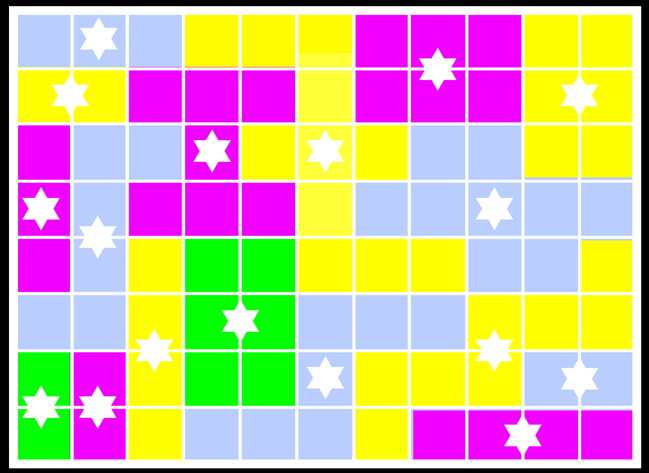

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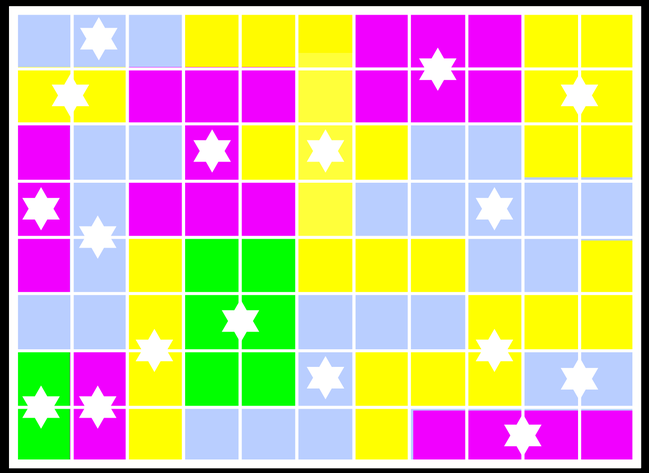

Also gives us a tiling using the solution galaxies as polyominoes

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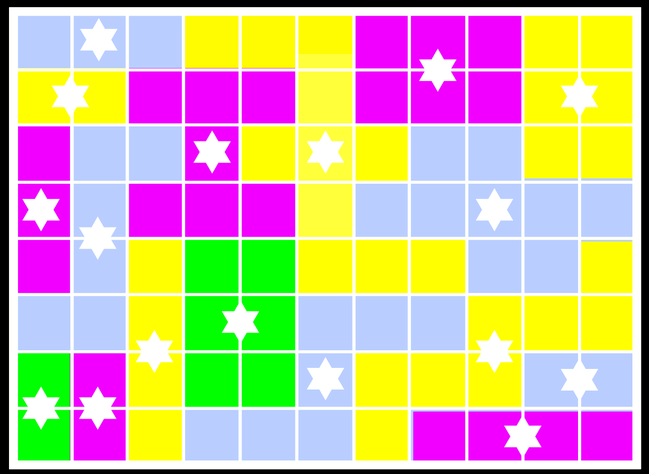
- ★ Do you want to solve some puzzles?

  http://www.nikoli.co.jp/en/puzzles/astronomical_show.html

  http://puzzlepicnic.com/genre?id=17

  https://www.gmpuzzles.com/blog/spiral-galaxies-rules-info/



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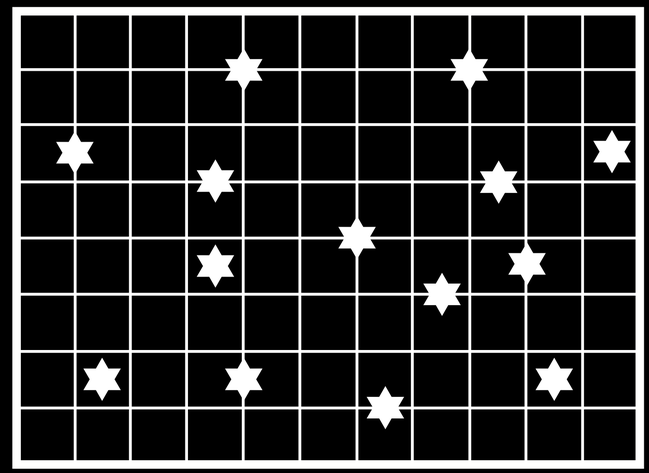

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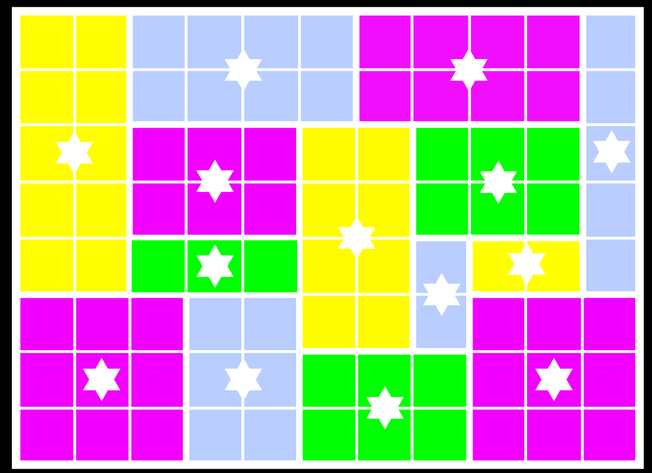

Generating Spiral Galaxies Puzzles

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- ⋆ What do we want to optimize when we generate a puzzle?

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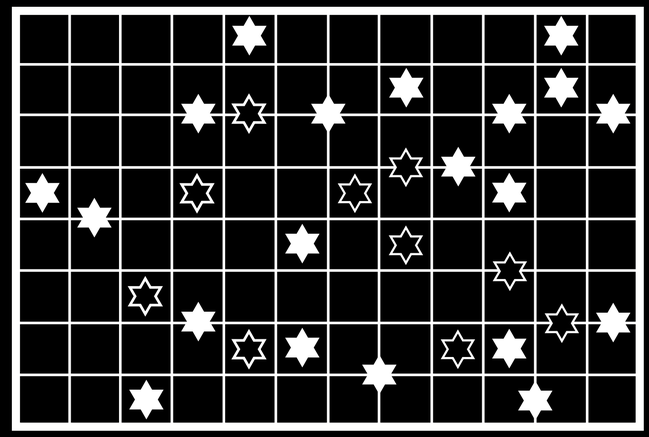

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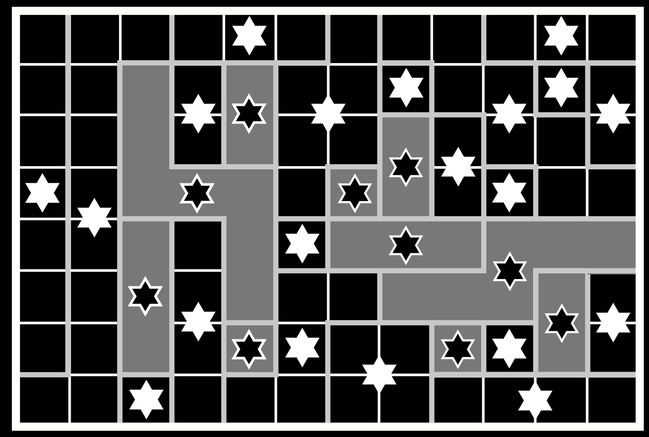

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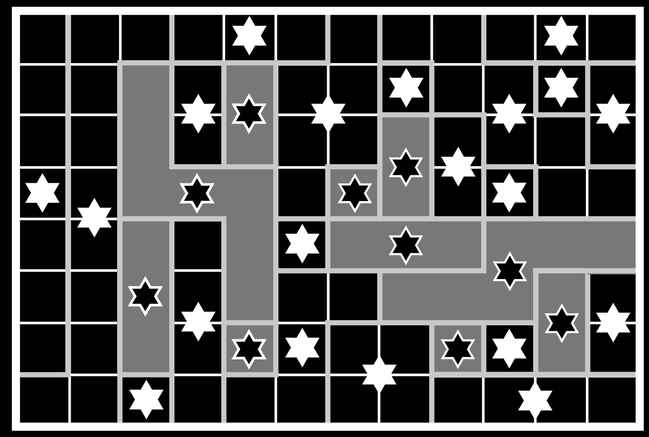

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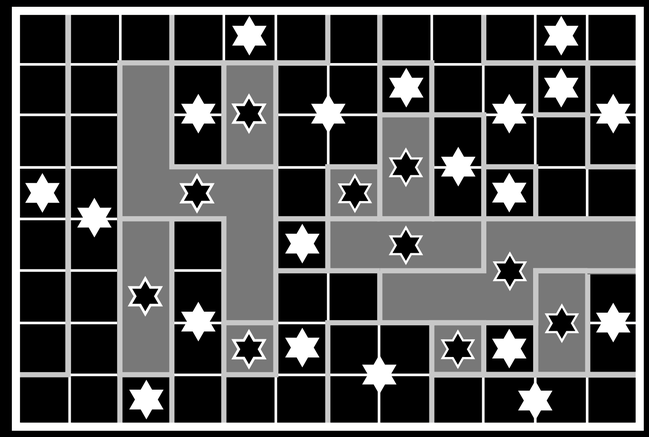
  - ➡ Minimum number of centers, such that there exist Spiral Galaxies that exactly cover a given shape

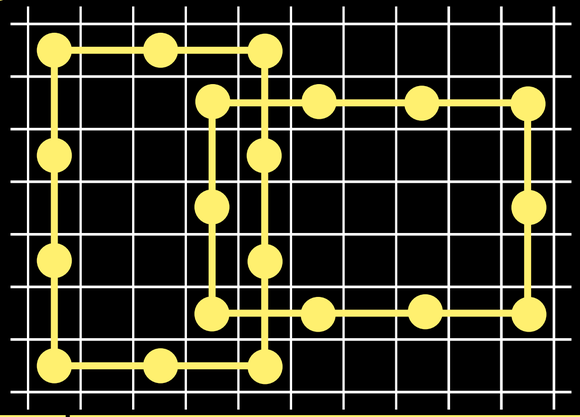


Results

- ⋆ Determining if a Spiral Galaxies board is solvable with only rectangular galaxies is NP-complete.
- ⋆ Determining if a Spiral Galaxies board is solvable with only 1x1, 1x3 and 3x1 galaxies is NP-complete and counting the number of solutions is #P-complete and ASP-complete.
- ⋆ Non-crossing matching in squared grid graphs is NP-complete.
- ⋆ Generating puzzles: Minimizing the number of centers on a Spiral Galaxies board, such that Spiral Galaxies with these centers exactly cover a given shape S is NP-complete.

Results

Squared grid graph:
Edges connect vertices at distance 2



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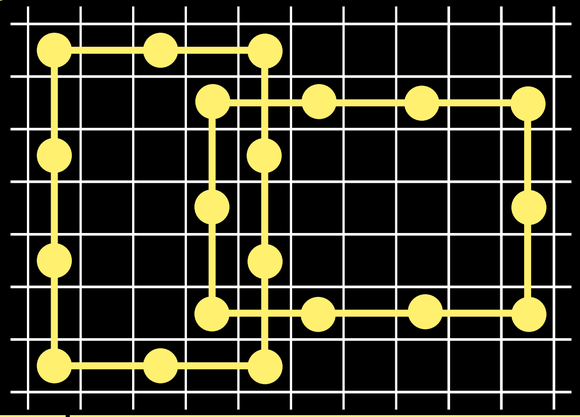

Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

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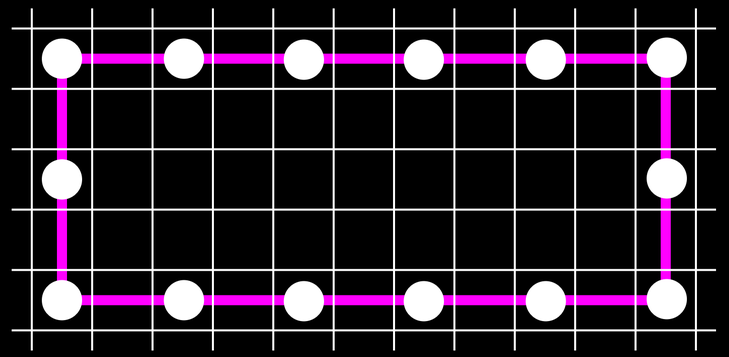
- ★ Disks with distance 2 can be connected by an edge



Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

- ★ Reduction from PLANAR 1-IN-3 SAT
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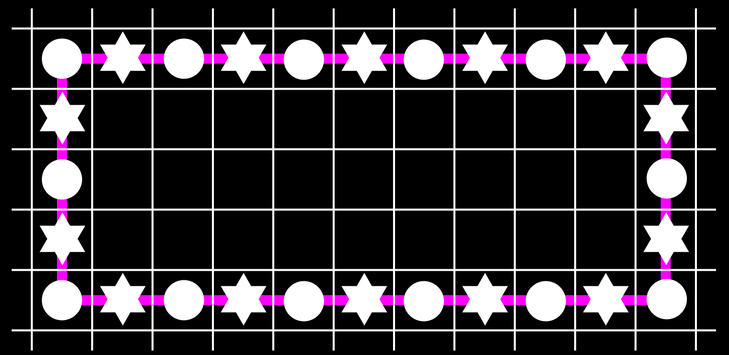
- ★ Disks with distance 2 can be connected by an edge
- ★ Centers ★ in middle of each potential edge



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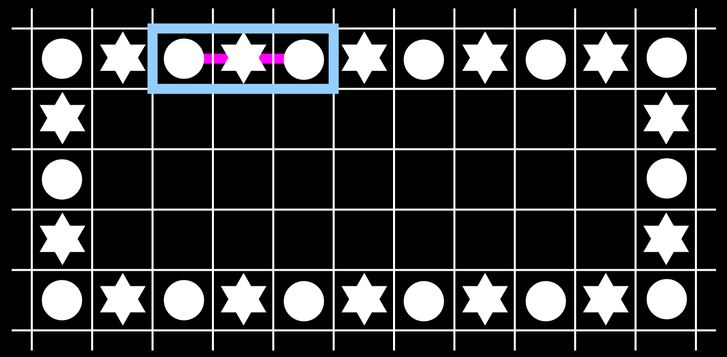
- ➡ 1x3 and 3x1 galaxies cover both disks ≙ **edge** between disks



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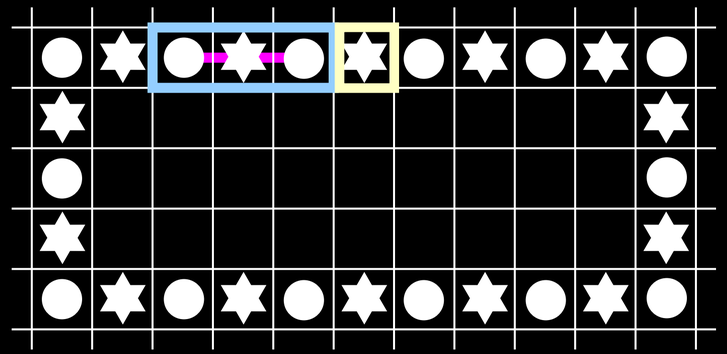
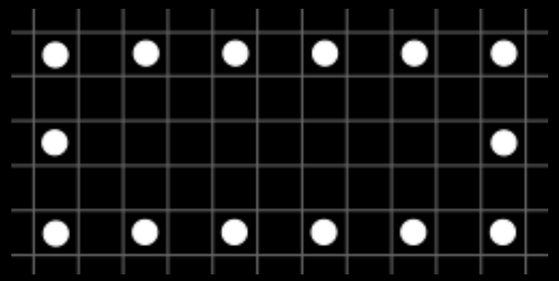
- ➡ 1x3 and 3x1 galaxies cover both disks ≙ **edge** between disks
- ➡ 1x1 galaxy does not cover disks ≙ **non-existing edge** between disks



Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

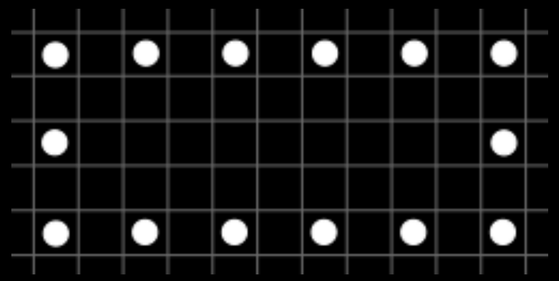
Variable loop

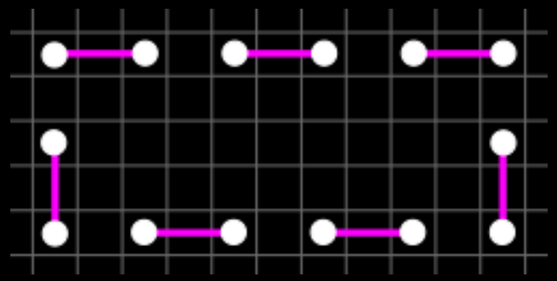


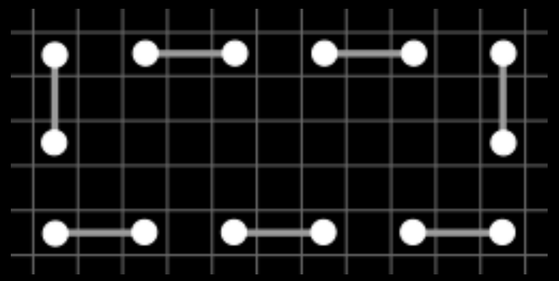
Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

Variable loop

Two possible states—“true” and “false”:

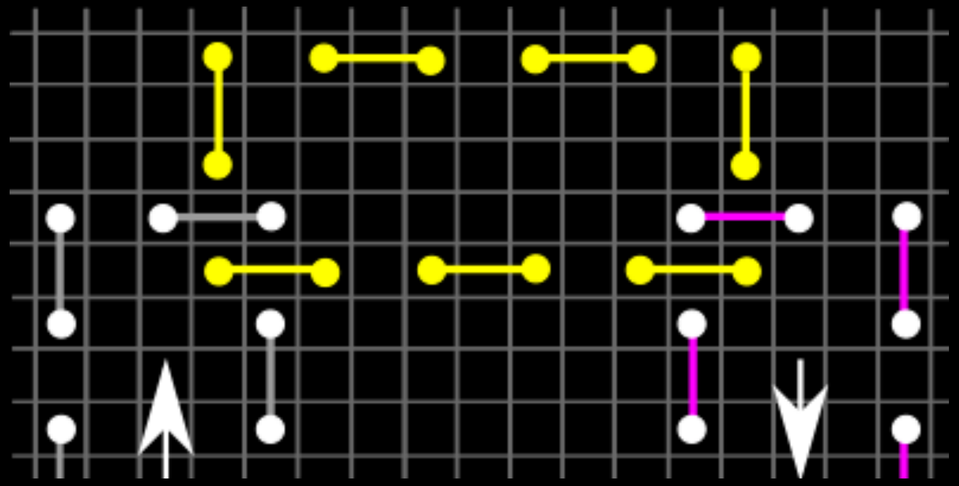


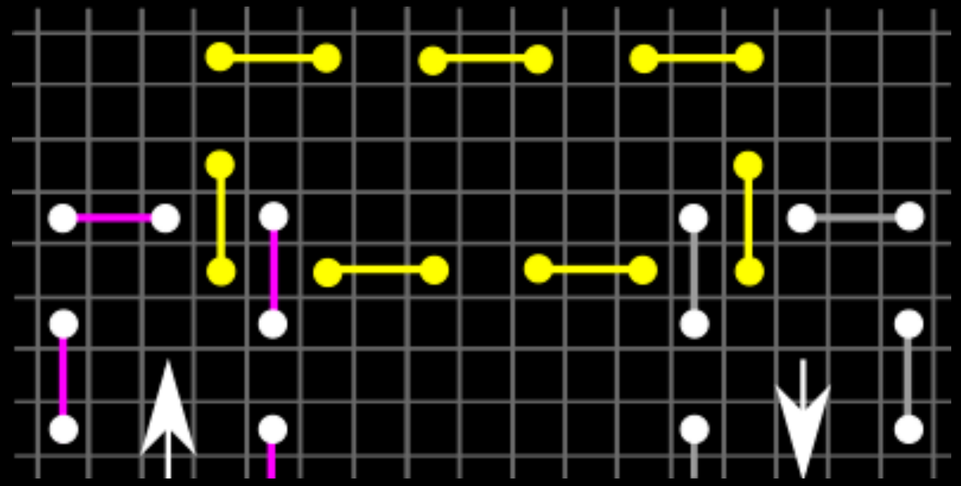




Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

Negation gadget

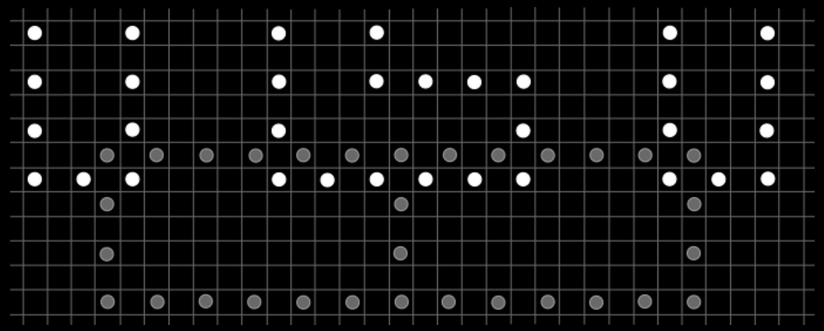




Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

Clause gadget

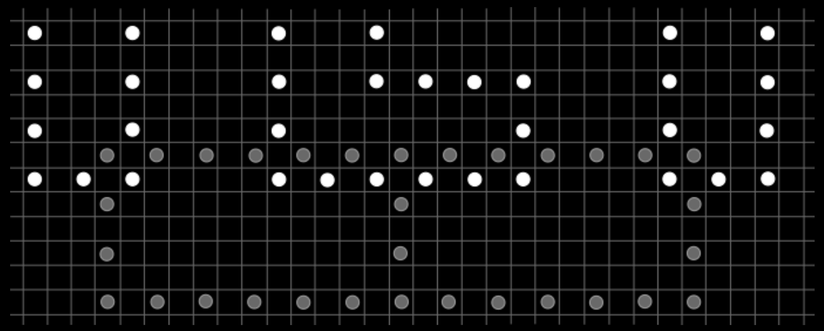
3 variable loops

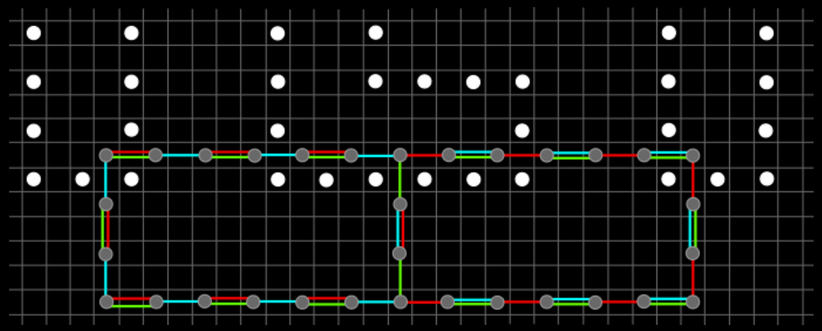


Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

Clause gadget

3 variable loops





Spiral Galaxies with 1x1, 1x3 and 3x1 Rectangles

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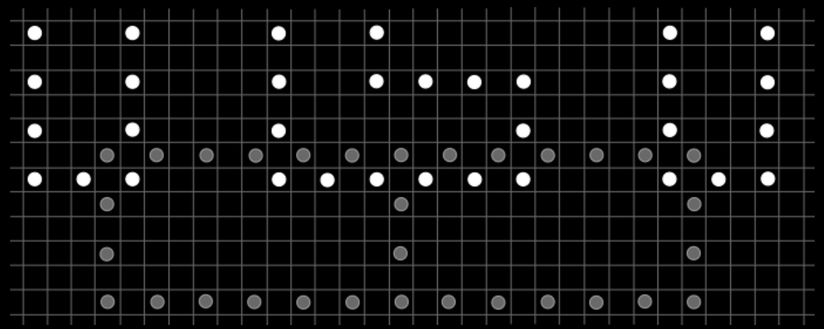
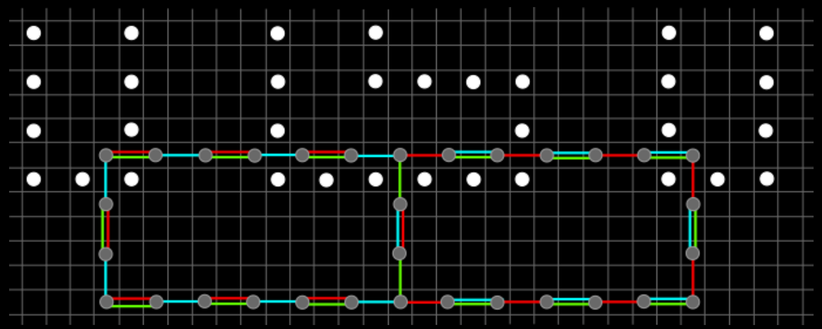
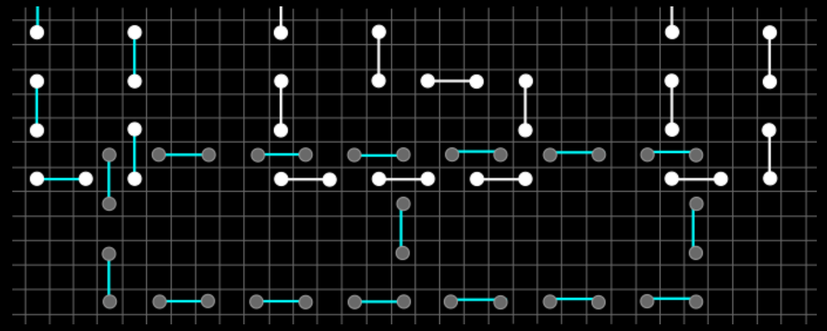
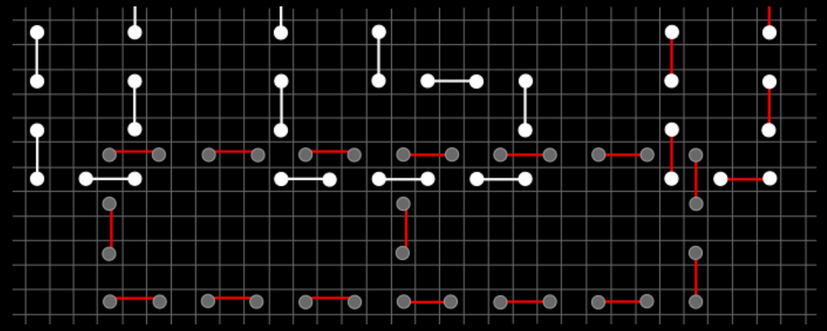
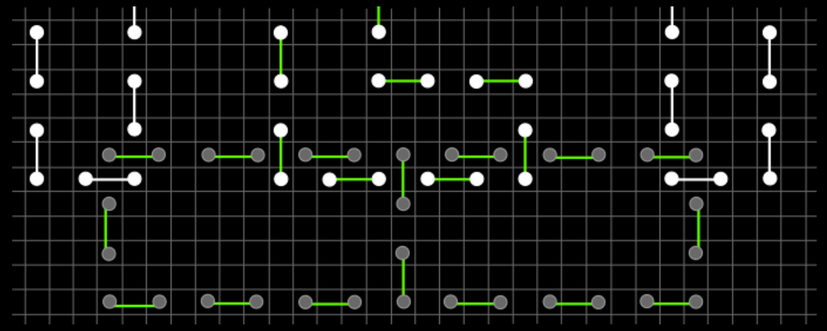


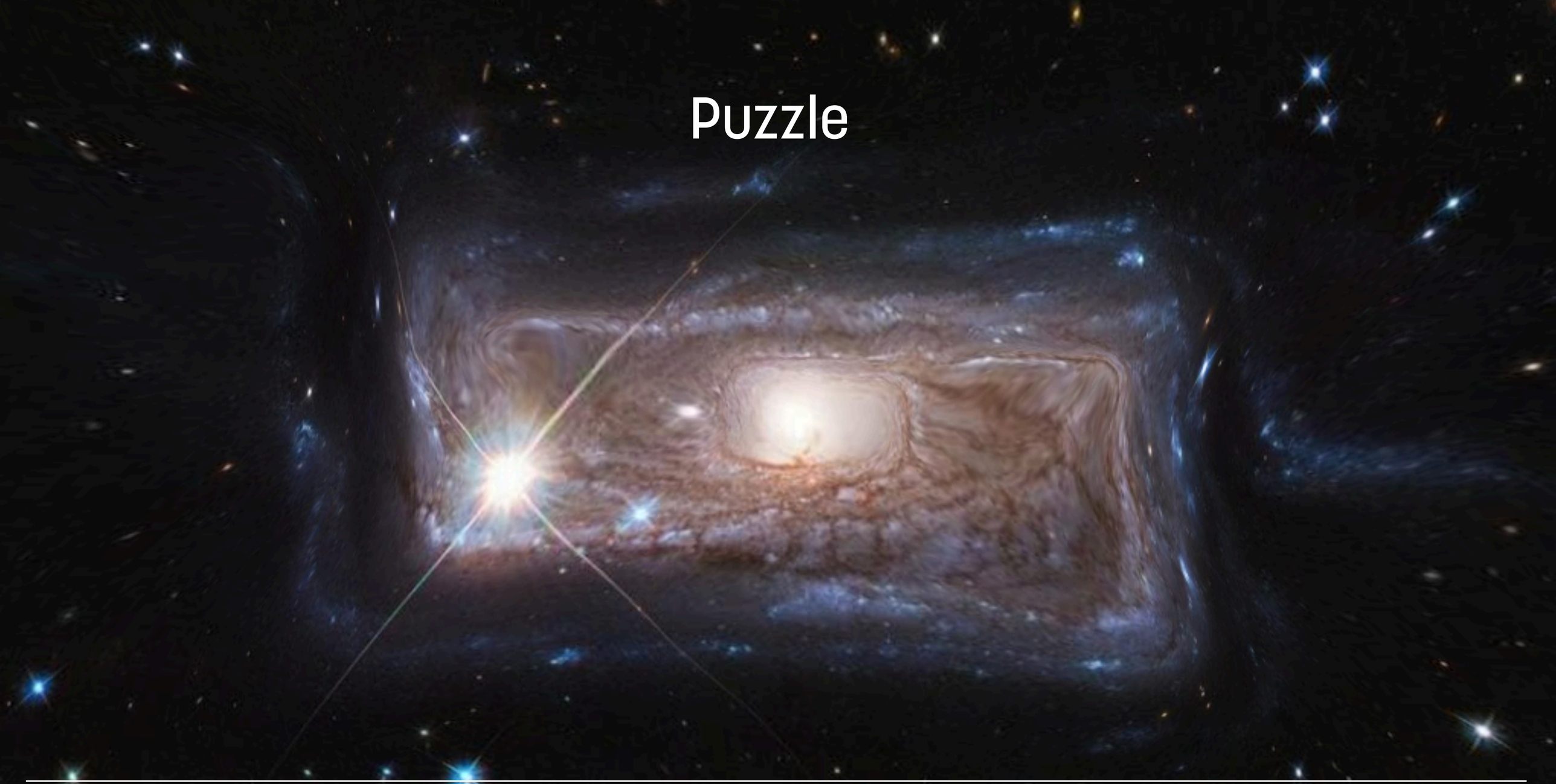




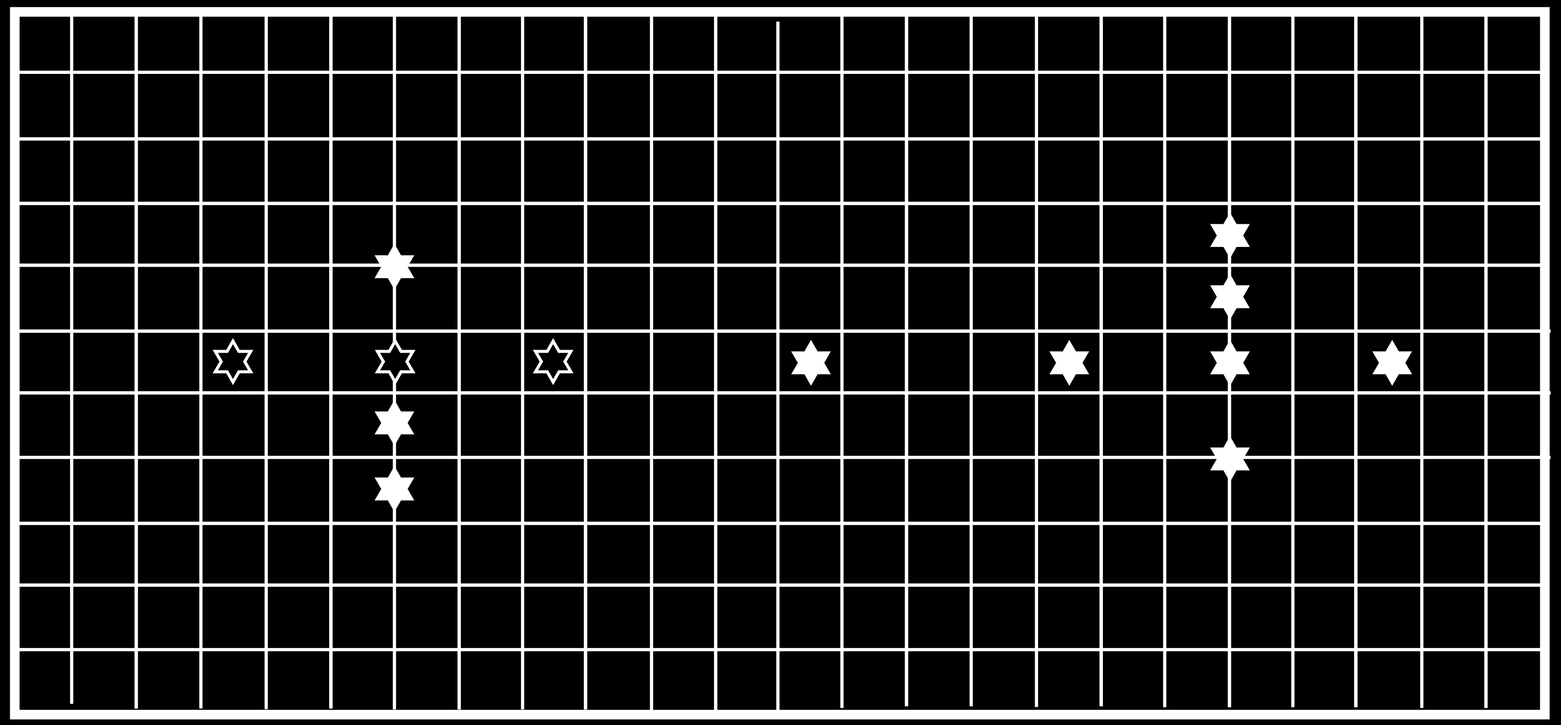




Puzzle



Can be solved for the letters A, B, H, P, R, S, Z (+E for disconnected galaxies)



SCAN ME


Puzzle? Look here:
SCAN ME
THANKS