Communications and Transport Systems
Department of Science and Technology **Fall 2019**
Linköping University

Christiane Schmidt

Design and Analysis of Algorithms Part 1 - Mathematical tools and Network problems homework 5, 23.01.2020

Problem 1 (Algorithm 8.7, Ford-Fulkerson):

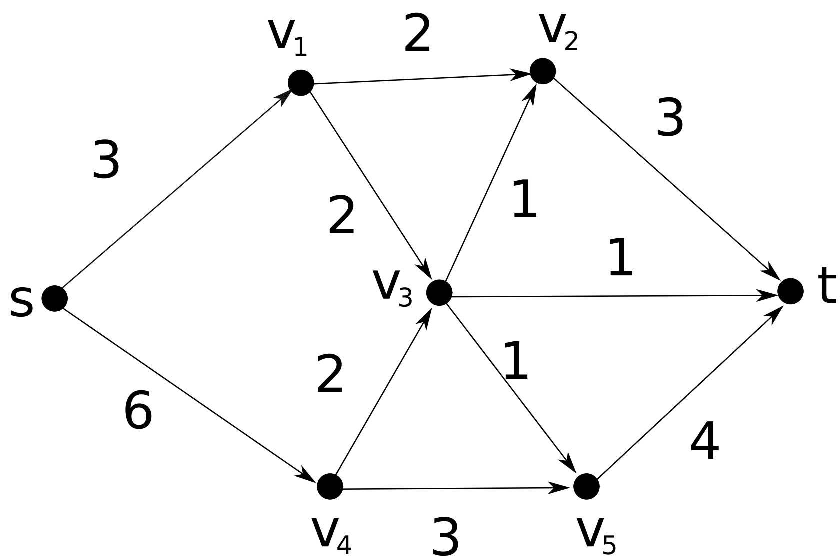

Figure 1: The network (G, u, s, t). The numbers at the edges give the capacities

Use the algorithm by Ford and Fulkerson to determine a maximum $s-t$-flow in the network (G, u, s, t). Give the residual graph in each step.
In addition: give a minimum cut.

Problem 2 (Menger's Theorem (Menger 1927)):
Two paths P and Q are called edge-disjoint if they have no common edge.
Let G be a graph (directed or undirected), let s and t be two vertices and $k \in \mathbb{N}$. Then there are k edge-disjoint s-t-paths if and only if after deleting $k-1$ edges t is still reachable from s.

Problem 3 (Ford-Fulkerson algorithm and irrational capacities):
Show that the algorithm by Ford and Fulkerson might not terminate when it is applied to a network with irrational capacities.

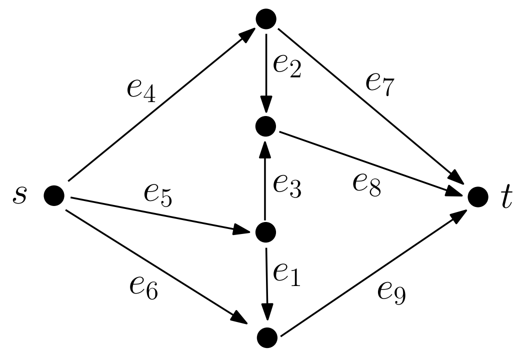

Figure 2: A network with irrational capacities

Consider the network in Figure 2 with capacities $u(e_1) = 1$, $u(e_2) = \sigma$, $u(e_3) = 1$ und $u(e_4) = u(e_5) = \ldots = u(e_9) = 4$, with $\sigma = \frac{\sqrt{5}-1}{2}$. First show $\sigma^n = \sigma^{n+1} + \sigma^{n+2}$.
(Hint: Consider the paths $P_1 = \{e_4, e_2, \overleftarrow{e_3}, e_1, e_9\}$, $P_2 = \{e_5, e_3, \overleftarrow{e_2}, e_7\}$, $P_3 = \{e_6, \overleftarrow{e_1}, e_3, e_8\}$ und $P_4 = \{e_5, e_3, e_8\}$. Show by induction that we can change the *residual capacities* of e_1, e_2 and e_3 from σ^n, σ^{n+1} and 0 to σ^{n+2}, σ^{n+3} and 0, respectively. Induction base: augment along P_4. Induction step: augment, consecutively, along P_1, P_2, P_1 and P_3.)

Problem 4 (Integer Flow): Show Corrolary 8.12 from the seminar: Let $N = (G, u, s, t)$ be a network. If the capacities $u(e)$ are all integers, then there exists a maximum flow in N, such that all $f(e)$ are integeres (in particular, the optimum flow is integer).

Problem 5 (PUSH-RELABEL algorithm): For each proof you can use all theorems, lemmata etc. with a smaller number.

(a) Show Proposition 8.20: During the execution of the Push-Relabel algorithm f is always an s-t-preflow and ψ is always a distance labeling with respect to f. (Hint: Show that the procedures PUSH and PRELABEL preserve these properties.)

(b) Show Lemma 8.21: If f is an s-t-preflow and ψ is a distance labeling with respect to f, then

   (1) s is reachable from any active vertex v in G_f.

   (2) t is not reachable from s in G_f.

   (Hint: For (1) consider the set of vertices that are reachable from an active vertex v. For (2) use contradiciton.)

(c) Show Theorem 8.22: When the algorithm 8.19 terminates, f is a maximum s-t-flow.

(d) Show Lemma 8.24: The number of saturating pushes is at most mn.

Problem 6 (MIN CUT problem): The MIN CUT problem is defined as follows:
INPUT: Network (G, u, s, t).
OUTPUT: An s-t-cut of minimum capacity.

Show how you can compute a MIN CUT in time $O(n^3)$.