**Communications and Transport Systems**
**Department of Science and Technology** **Fall 2021**
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# Design and Analysis of Algorithms Part 1 - Mathematical Tools and Network Problems homework 1, 03.11.2021

**Problem 1 (Graphs):**

(a) Show $\sum_{i=1}^{n} |\delta(v_i)| = 2m$ for *all* graphs $G$ with $n$ vertices and $m$ edges.

(b) Let $H$ be a complete graph with $n$ vertices. Show that the number of edges in $H$ equals $\frac{n}{2}(n-1)$.

**Problem 2 (Connected graphs):**

(a) Let $G$ be a graph with $n$ vertices and assume that each vertex of $G$ has degree at least $(n-1)/2$. Show that $G$ must be connected.

(b) Show: A graph $G$ is connected if and only if there exists an edge $e = \{v, w\}$ with $v \in V_1$ and $w \in V_2$ whenever $V(G) = V_1 \cupdot V_2$ (i.e., $V_1 \cap V_2 = \emptyset$) .

(c) Show: If $G$ is not connected, the complementary graph $\overline{G}$ is connected.

(d) Show: A connected graph with $n$ vertices has at least $n-1$ edges.

**Problem 3 (Cuts):**

Show: for a digraph $G$ and any two sets $X, Y \subseteq V(G)$:

(a) $|\delta^+(X)| + |\delta^+(Y)| = |\delta^+(X \cap Y)| + |\delta^+(X \cup Y)| + |E^+(X,Y)| + |E^+(Y,X)|$.

(b) $|\delta^-(X)| + |\delta^-(Y)| = |\delta^-(X \cap Y)| + |\delta^-(X \cup Y)| + |E^+(X,Y)| + |E^+(Y,X)|$.

For an undirected graph $G$ and any two sets $X, Y \subseteq V(G)$:

(c) $|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X,Y)|$.

(d) $|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)|$.

**Problem 4 (O-Notation):**

(a) For the following functions find the constants $c$ (or $c_1$ and $c_2$) and $n_0$ and show with help of these constants that the given function is in the given class.

$$\begin{aligned} f_1(n) &= \frac{n^{14}}{4^n} \in O(1) \\ f_2(n) &= 2n^2 + 3n + 1 \in O(n^3) \\ f_3(n) &= \sum_{i=1}^{n} i \in \Theta(n^2) \end{aligned}$$

(b) Show: Let $f, g : \mathbb{N} \mapsto \mathbb{R}$ be two functions; then the following statements hold:

(i) $f \in \Theta(g) \Leftrightarrow g \in \Theta(f)$

(ii) $f \in \Theta(g) \Leftrightarrow f \in O(g)$ und $f \in \Omega(g)$

(iii) $f \in O(g) \Leftrightarrow g \in \Omega(f)$