Communications and Transport Systems
Department of Science and Technology **Spring 2017**
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Design and Analysis of Algorithms Part 1 - Mathematical tools and Network problems homework 1, 09.03.2017

Problem 1 (Graphs):

(a) Show $\sum_{i=1}^{n} \delta(v_i) = 2m$ for *all* graphs G with n vertices and m edges.

(b) Let H be a complete graph with n vertices. Show that the number of edges in H equals $\frac{n}{2}(n-1)$.

Problem 2 (Connected graphs):

(a) Let G be a graph with n vertices and assume that each vertex of G has degree at least $(n-1)/2$. Show that G must be connected.

(b) Show: A graph G is connected if and only if there exists an edge $e = \{v, w\}$ with $v \in V_1$ and $w \in V_2$ whenever $V(G) = V_1 \mathbin{\dot\cup} V_2$ (i.e., $V_1 \cap V_2 = \varnothing$) .

(c) Show: If G is not connected, the complementary graph $\overline{G}$ is connected.

(d) Show: A connected graph with n vertices has at least $n-1$ edges.

Problem 3 (Cuts):

Show: for a digraph G and any two sets $X, Y \subseteq V(G)$:

(a) $|\delta^+(X)| + |\delta^+(Y)| = |\delta^+(X \cap Y)| + |\delta^+(X \cup Y)| + |E^+(X,Y)| + |E^+(Y,X)|$.

(b) $|\delta^-(X)| + |\delta^-(Y)| = |\delta^-(X \cap Y)| + |\delta^-(X \cup Y)| + |E^+(X,Y)| + |E^+(Y,X)|$.

For an undirected graph G and any two sets $X, Y \subseteq V(G)$:

(c) $|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X,Y)|$.

(d) $|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)|$.

Problem 4 (Trees - moved to Homework set 2):

Problem 5 (Directed cycles and directed cuts - moved to Homework set 2):

Problem 6 (O-Notation):

(a) For the following functions find the constants c (or c_1 and c_2) and n_0 and show with help of these constants that the given function is in the given class.

$$\begin{aligned} f_1(n) &= \frac{n^{14}}{4^n} && \in O(1) \\ f_2(n) &= 2n^2 + 3n + 1 && \in O(n^3) \\ f_3(n) &= \sum_{i=1}^{n} i && \in \Theta(n^2) \end{aligned}$$

(b) Show: Let $f, g : \mathbb{N} \mapsto \mathbb{R}$ be two functions; then the following statements hold:

(i) $f \in \Theta(g) \Leftrightarrow g \in \Theta(f)$

(ii) $f \in \Theta(g) \Leftrightarrow f \in O(g)$ und $f \in \Omega(g)$

(iii) $f \in O(g) \Leftrightarrow g \in \Omega(f)$

Problem 7 (Best-case running time for quicksort):
Proof Lemma 3.12 from the lecture, that is, the best-case running time for quicksort

Problem 8 (Heap Sort):
Prepare a 10 minute presentation of heapsort: the algorithm, its correctness and runnning time.

Problem 9 (Merge sort):

Sort the sequence $(33, 14, 7, 9, 2, 11, 45, 21)$ using merge sort. Give the intermediate steps in appropriate form.

Problem 10 (Mastertheorem):

a) Determine the asymptotic growth of the following recursion using the master theorem

$U(n) = 4 \cdot U(\frac{n}{3}) + 17 \cdot n^2 + 20 \cdot U(\frac{n}{6})$.

Determine the value of all parameters used in the master theorem.

b) Determine the asymptotic growth of the following recursion using the master theorem

$V(n) = 14 \cdot V(\frac{n}{36}) + 23n + 12 \cdot V(\frac{n}{24}) + V(\frac{n}{10})$.

Determine the value of all parameters used in the master theorem.

c) Determine the asymptotic growth of the following recursion using the master theorem

$T(n) = 49 \cdot T(\frac{n}{7}) + 42n$.

Determine the value of all parameters used in the master theorem.

Problem 11 (Quicksort):

Sort the numbers in the following array using the algorithm quicksort presented in the lecture.

$$A[1] = 14 \qquad A[2] = 3 \qquad A[3] = 7 \qquad A[4] = 1 \qquad A[5] = 2$$

The reference element should be chosen as in the lecture (that is, $A[r]$). Give the array after **each** swap operation. Give the intermediate steps from Quicksort- and Partition calls.

Problem 12 (Data structures for graphs - added problem):
Take a look at slide 100 of the reordered lecture, that is DAA-1-new.pdf. Write the adjacency matrix, the incidence matrix, the adjacency list, and the edge list for the graph G in Figure 1.

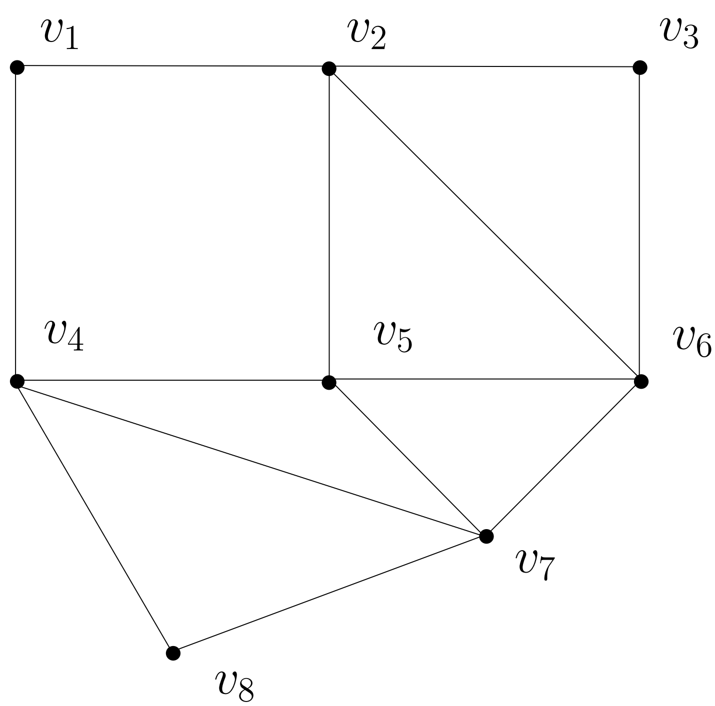

Figure 1: A graph G.