Finding Line Intersection in 3D

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The intersection of two lines in 3D space is hard to come by. Both lines must lie in the exact same plane which may not be the case. Here follow the derivation of the closest point between two lines, which can be regarded as the intersection as long as the distance is smaller than a pre-determined value.

$$
\begin{aligned}
\vec{L}_0(t) &= \vec{p}_0 + t_0\vec{v}_0 \\
\vec{L}_1(t) &= \vec{p}_1 + t_1\vec{v}_1 \\
\Delta &= |\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0| \\
\frac{\partial}{\partial t_0}\Delta &= \frac{\partial}{\partial t_0}\,|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0| \\
\frac{\partial}{\partial t_1}\Delta &= \frac{\partial}{\partial t_1}\,|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0|
\end{aligned}
$$

$$
\left\{
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\left|t\vec{a}+\vec{b}\right| &= \tfrac{\mathrm{d}}{\mathrm{d}t}\sqrt{\left(ta_x+b_x\right)^2+\left(ta_y+b_y\right)^2} \\
&= \tfrac{2(ta_x+b_x)a_x+2(ta_y+b_y)a_y}{\left|t\vec{a}+\vec{b}\right|} \\
&= \tfrac{2\vec{a}\cdot\left(t\vec{a}+\vec{b}\right)}{\left|t\vec{a}+\vec{b}\right|}
\end{aligned}
\right\}
\tag{1}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t_0}\Delta = 0 &= \frac{\partial}{\partial t_0}\,|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0| \\
0 &= \frac{2\vec{v}_0\cdot(\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0)}{|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0|} \\
0 &= \vec{v}_0\cdot(\vec{p}_1 - \vec{p}_0) + t_1\vec{v}_0\cdot\vec{v}_1 - t_0\vec{v}_0\cdot\vec{v}_0 \\
t_0\vec{v}_0\cdot\vec{v}_0 - t_1\vec{v}_0\cdot\vec{v}_1 &= \vec{v}_0\cdot(\vec{p}_1 - \vec{p}_0)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t_1}\Delta = 0 &= \frac{\partial}{\partial t_1}\,|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0| \\
0 &= \frac{2\vec{v}_1\cdot(\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0)}{|\vec{p}_1 - \vec{p}_0 + t_1\vec{v}_1 - t_0\vec{v}_0|} \\
0 &= \vec{v}_1\cdot(\vec{p}_1 - \vec{p}_0) + t_1\vec{v}_1\cdot\vec{v}_1 - t_0\vec{v}_1\cdot\vec{v}_0 \\
t_0\vec{v}_1\cdot\vec{v}_0 - t_1\vec{v}_1\cdot\vec{v}_1 &= \vec{v}_1\cdot(\vec{p}_1 - \vec{p}_0)
\end{aligned}
$$

$$
\begin{pmatrix} \vec{v}_0\cdot\vec{v}_0 & -\vec{v}_0\cdot\vec{v}_1 \\ \vec{v}_1\cdot\vec{v}_0 & -\vec{v}_1\cdot\vec{v}_1 \end{pmatrix}
\begin{pmatrix} t_0 \\ t_1 \end{pmatrix}
=
\begin{pmatrix} \vec{v}_0\cdot(\vec{p}_1 - \vec{p}_0) \\ \vec{v}_1\cdot(\vec{p}_1 - \vec{p}_0) \end{pmatrix}
\tag{2}
$$