# Signal Reconstruction Performance Under Quantized Noisy Compressed Sensing

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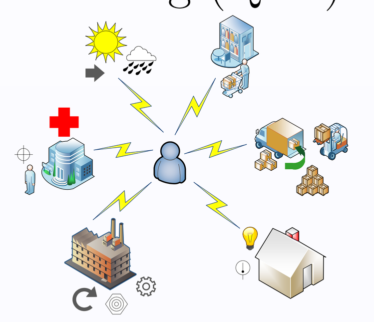
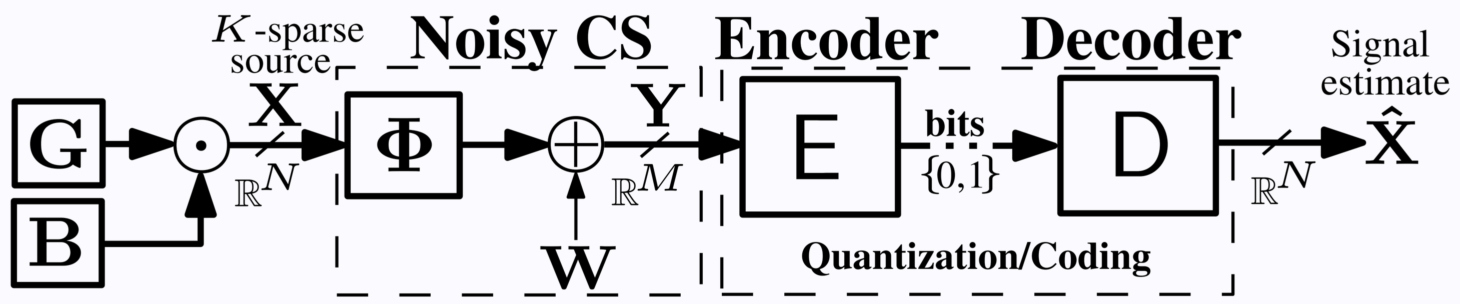
## Abstract

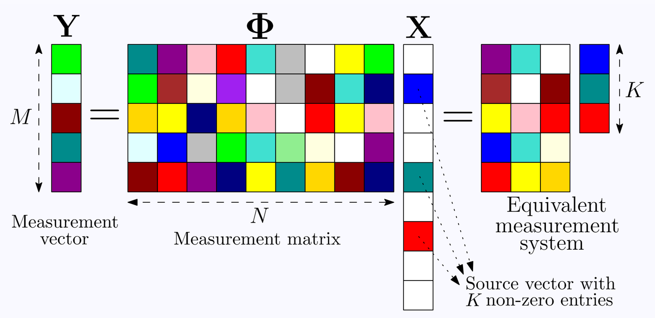

- Finite-rate acquisition of sparse sources via noisy compressive measurements
- Motivated by low-power sensor and IoT applications
  - Minimizing the number of bits to compress and communicate real-valued sources with a pre-defined distortion is crucial to prolong the network lifetime etc.
- Rate-distortion (RD) performance of quantized compressed sensing (QCS)
- Practical QCS methods: symbol-by-symbol quantizers and three different compression strategies
- Information-theoretical QCS limits through a lower bound and 2) numerical approximation of the remote RD function (RDF)

## QCS Framework




- The encoder E observes the information source $\mathbf{X}$ via noisy linear measurements $\mathbf{Y}$ $\Rightarrow$ *Remote source coding*
- Finite-rate quantization/encoding at E
- Signal reconstruction at decoder D
- Problem: Study the achievable signal reconstruction performance of 1) *Practical* QCS methods where E-D is a **symbol-by-symbol** quantizer and relies on different compression strategies 2) *Information-theoretical* context with excessively high complex E-D



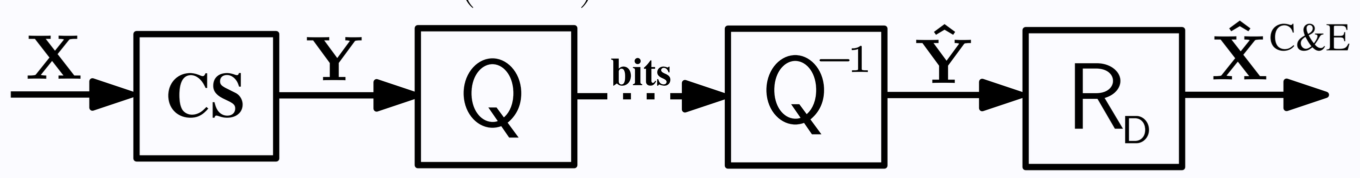

- Distortion: Mean square error (MSE) $D = \mathbb{E}[\|\mathbf{X} - \hat{\mathbf{X}}\|_2^2]$
- Signal: $K$-sparse source $\mathbf{X}_n = [X_{n,1} \cdots X_{n,N}]^{\mathrm{T}}$, $\|\mathbf{x}\|_0 = K \leq N$
  - $\mathbf{X}_n = \mathbf{G}_n \odot \mathbf{B}_n$: 1) Gaussian $\mathbf{G}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_G})$; 2) binary $\mathbf{B}_n$, $\|\mathbf{b}_s\|_0 = K$, $s = 1, \ldots, \binom{N}{K}$
- Sensing: Indirect observations by a CS-based sensor: $\mathbf{Y}_n = \mathbf{\Phi}\mathbf{X}_n + \mathbf{W}_n$
  - Fixed and known measurement matrix $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$, $K \leq M \leq N$
  - Noise random vector $\mathbf{W}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_W})$

## Quantizers

- Quantizer Q with 1) encoder regions $\mathcal{S}_i \subseteq \mathbb{R}^L$, $\bigcup_{i=1}^{|\mathcal{I}|} \mathcal{S}_i = \mathbb{R}^L$, and 2) a reconstruction codebook $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{|\mathcal{I}|}\}$ with $\mathbf{c}_i \in \mathbb{R}^L$
  - Quantizer encoder $\mathsf{Q} : \mathbb{R}^L \to \mathcal{I}$ operating as $i = \mathsf{Q}(\mathbf{u})$ if $\mathbf{u} \in \mathcal{S}_i$
  - Quantizer decoder $\mathsf{Q}^{-1} : \mathcal{I} \to \mathcal{C}$ operating as $\hat{\mathbf{u}} = \mathsf{Q}^{-1}(i) = \mathbf{c}_i \in \mathcal{C}$
- 1) **A uniform scalar quantizer** (USQ) $\mathsf{Q}^{\mathrm{u}}$ of rate $R'$ bits/$U$
- 2) **A fixed-rate Lloyd-Max quantizer** $\mathsf{Q}^{\mathrm{lm}}$ of rate $R'$ bits/$\mathbf{U}$ optimizes the $|\mathcal{I}| = 2^{R'}$ quantization regions and the codebook to minimize the MSE distortion $D' = \sum_{i=1}^{2^{R'}} p(i)\mathbb{E}[\|\mathbf{U} - \mathbf{c}_i^{\mathrm{lm}}\|_2^2 | I = i]$
  - Two-step iterative Lloyd and Linde-Buzo-Gray (LBG) algorithms for offline training
- 3) **A variable-rate quantizer** $\mathsf{Q}^{\mathrm{vr}}$ minimizes a cost function $(1-\mu')D' - \mu' \sum_{i \in \mathcal{I}} p(i)\log(p(i))$ for an RD trade-off parameter $\mu' \in [0,1]$
  - Three-step training algorithm using **entropy-constrained quantization** (ECSQ/ECVQ)
  - Source codebook $\mathcal{H} \triangleq \{\mathbf{h}_1, \ldots, \mathbf{h}_{|\mathcal{H}|}\}$ with *variable-length* binary codewords and average rate $\sum_{i=1}^{|\mathcal{H}|} p(i)l(\mathbf{h}_i)$ bits/$\mathbf{U}$ (e.g. Huffman coding)

## Practical QCS Methods

- Three compression strategies of different complexity-performance tradeoffs
- **A Compress-and-Estimate** (C&E) scheme:

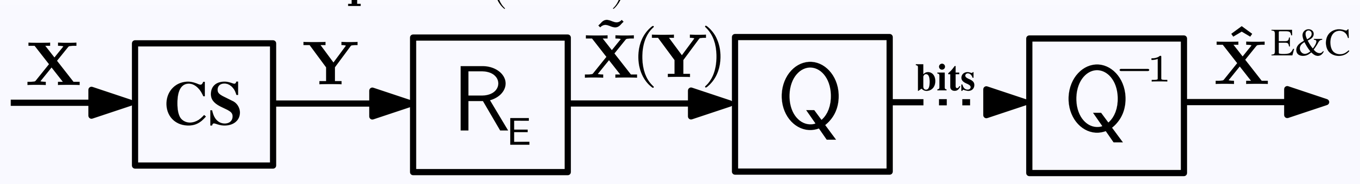



1) A compression stage: the encoder quantizes $\mathbf{Y}$ with respect to an MSE distortion that depends only on $\mathbf{Y}$ (not on $\mathbf{X}$)
2) An estimation stage: the decoder estimates $\mathbf{X}$ from the decoded quantized measurements $\hat{\mathbf{Y}} = \mathsf{Q}^{-1}(\mathsf{Q}(\mathbf{Y}))$

$$D^{\mathrm{C\&E}} = \mathbb{E}\big[\|\mathbf{X} - \hat{\mathbf{X}}^{\mathrm{C\&E}}\|_2^2\big] = \mathbb{E}\big[\|\mathbf{X} - \mathsf{R}_{\mathrm{D}}[\mathsf{Q}^{-1}(\mathsf{Q}(\mathbf{Y}))]\|_2^2\big]$$

– $\mathsf{R}_{\mathrm{D}} : \hat{\mathbf{Y}} \to \hat{\mathbf{X}}^{\mathrm{C\&E}}$ is a signal reconstruction algorithm at D
- C&E principle underlies many early QCS algorithms

- **An Estimate-and-Compress** (E&C) scheme:

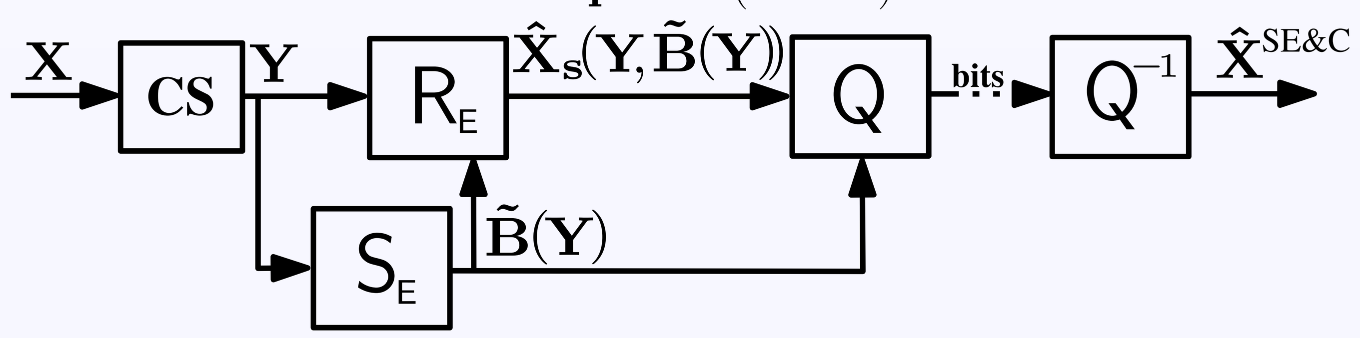



1) An estimation stage: the encoder estimates $\mathbf{X}$ from $\mathbf{Y}$
2) A compression stage: the encoder quantizes the resulting estimate with respect to an MSE distortion of $\mathbf{X}$

$$D^{\mathrm{E\&C}} = \mathbb{E}\big[\|\mathbf{X} - \hat{\mathbf{X}}^{\mathrm{E\&C}}\|_2^2\big] = \mathbb{E}\big[\|\mathbf{X} - \mathsf{Q}^{-1}[\mathsf{Q}(\mathsf{R}_{\mathrm{E}}(\mathbf{Y}))]\|_2^2\big]$$

– $\mathsf{R}_{\mathrm{E}} : \mathbf{Y} \to \tilde{\mathbf{X}}(\mathbf{Y})$ is a reconstruction algorithm at E
– $\tilde{\mathbf{X}}(\mathbf{Y})$ denotes the estimator of $\mathbf{X}$ from $\mathbf{Y}$
- E&C is *the optimal encoding structure* for remote source coding

- **A Support-Estimation-and-Compress** (SE&C) scheme:



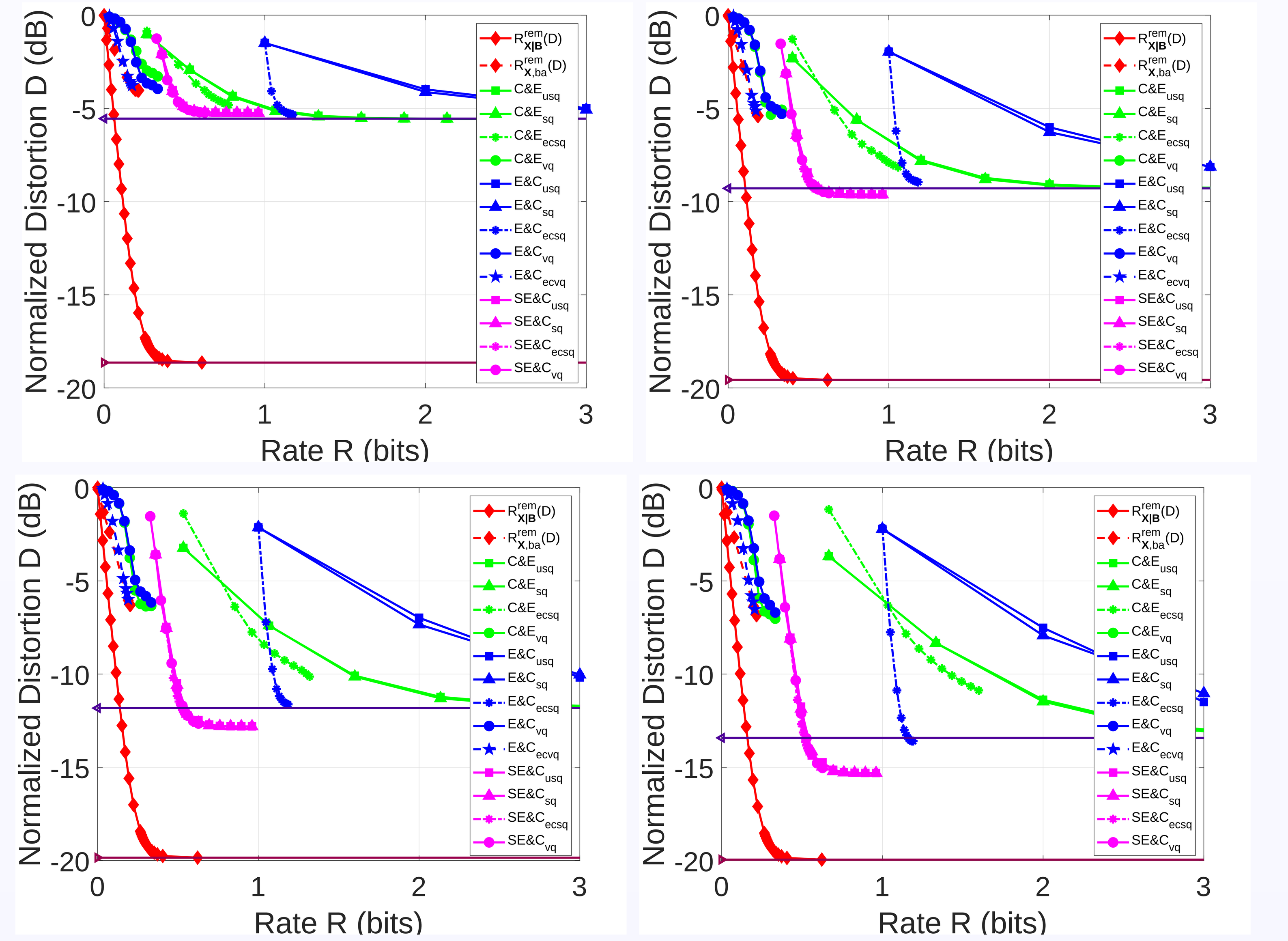

1) A support estimation stage: the encoder estimates $\mathbf{B}$ from $\mathbf{Y}$
2) An estimation stage: the encoder estimates $\mathbf{X}$ given $\mathbf{Y}$ and the support estimator
3) A two-phase compression stage: the encoder compresses the estimate by i) Lossless compression for $\hat{\mathbf{B}}$, ii) Lossy compression for the non-zero part

$$D^{\mathrm{SE\&C}} = \mathbb{E}\big[\|\mathbf{X} - \hat{\mathbf{X}}^{\mathrm{SE\&C}}\|_2^2\big] = \mathbb{E}\big[\|\mathbf{X} - \mathsf{Q}^{-1}[\mathsf{Q}(\mathsf{S}_{\mathrm{E}}(\mathbf{Y}) + \mathsf{R}_{\mathrm{E}}(\mathbf{Y}|\mathsf{S}_{\mathrm{E}}(\mathbf{Y})))]\|_2^2\big]$$

– $\mathsf{S}_{\mathrm{E}} : \mathbf{Y} \to \tilde{\mathbf{B}}(\mathbf{Y})$ is a support estimation algorithm at E
– $\tilde{\mathbf{B}}(\mathbf{Y})$ denotes the estimator of $\mathbf{B}$ from $\mathbf{Y}$
- The sparsity $K$ is assumed to be known by $\mathsf{S}_{\mathrm{E}}(\cdot)$

## Numerical Results

- QCS algorithms with 1) USQ, 2) SQ, 3) ECSQ, 4) VQ, and 5) ECVQ
- The compression limit of QCS: the minimum achievable rate $R$ for a given distortion $D$ is given by *the remote RDF* of source $\mathbf{X}$, $R_{\mathbf{X}}^{\mathrm{rem}}(D)$
  - 1) Analytical lower bound: the conditional remote RDF $R_{\mathbf{X}|\mathbf{B}}^{\mathrm{rem}}(D) \leq R_{\mathbf{X}}^{\mathrm{rem}}(D)$ that assumes support side information $\mathbf{B}$ at the encoder and decoder
  - 2) Numerical approximation of $R_{\mathbf{X}}^{\mathrm{rem}}(D)$ via the modified Blahut-Arimoto (BA) algorithm $R_{\mathbf{X},\mathrm{ba}}^{\mathrm{rem}}(D) \simeq R_{\mathbf{X}}^{\mathrm{rem}}(D)$ under VQ discretized encoder input $\mathbf{Y}$
- An example with $N = 30$, $M = \{8, 12, 16, 20\}$, $K = 2$
  - $\mathbf{\Sigma_G} = \boldsymbol{I}_N$, $\mathbf{\Sigma_W} = 0.01\boldsymbol{I}_M$, $p(\mathbf{b}_s) = 1/|\mathcal{B}|$, $\forall \mathbf{b}_s \in \mathcal{B}$, and DCT-type $\mathbf{\Phi}$
  - Basis pursuit denoising (BPDN) as $\mathsf{R}_{\mathrm{D}}$ in C&E, $\mathsf{R}_{\mathrm{E}}$ in E&C, and $\mathsf{S}_{\mathrm{E}}$ in SE&C ($\mathsf{S}_{\mathrm{E}}$ preserves only the indices of the $K$ largest magnitudes)
  - Horizontal lines: the analytical MMSE estimation floor with support side information (lower); the error floor of the BPDN reconstruction (upper)
- SQ based: E&C < C&E < SE&C; VQ based: SE&C < C&E < E&C




## Acknowledgments & References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  - M. Leinonen, M. Codreanu, and M. Juntti, *"Distributed Distortion-Rate Optimized Compressed Sensing in Wireless Sensor Networks"*, IEEE Transactions on Communications, Vol. 66, pp. 1609 - 1623, Apr. 2018.