An Exact Expression for the Average AoI in a Multi-Source M/M/1 Queueing Model

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Background: Definition and Appliances

- Time sensitive information updates of a random process
  - Temperature of a specific environment (room, greenhouse, etc.)
  - A vehicular status (position, acceleration, etc.)
- A status update packet contains
  - The measured value of the monitored process
  - A time stamp representing the time when the sample was generated

One key enabler for these services is the freshness of the sensor’s information at the destinations.
Background: Age of Information

- The traditional metrics (throughput and delay) can not fully characterize the information freshness.
- AoI is the time elapsed since the last received status update was generated, i.e., the random process

\[ \Delta(t) = t - u(t) \]  \hspace{1cm} (1)

- \( u(t) \) is the time stamp of the most recently received update.

- Average AoI: The most commonly used metrics for evaluating the AoI.
System Model

- A set of independent sources
- The packets are generated according to the Poisson process
- An exponentially distributed service time (1/\( \mu \))
- FCFS multi-source M/M/1 queueing model
- Two sources without loss of generality
- The average AoI of source 1

\[
\Delta_1 = \lambda_1 \left( \frac{\mathbb{E}[X_{1,i}^2]}{2} + \mathbb{E}[X_{1,i}S_{1,i}] + \mathbb{E}[X_{1,i}W_{1,i}] \right) 
\]  (2)

- \( X_{1,i} \) represents the \( i \)th interarrival time of source 1
- \( S_{1,i} \) represents the service time of packet 1, \( i \)
- \( W_{1,i} \) represents the waiting time of packet 1, \( i \)

Aol in a Multi-Source M/M/1 Queueing Model (1/9)

- The first term in (2): The interarrival time of source 1 follows the exponential distribution with parameter $\lambda_1$
  \[ \mathbb{E}[X_{1,i}^2] = 2/\lambda_1^2 \]  
  \[ (3) \]

- The second term in (2): The interarrival time and service time of the packet $1, i$ are independent
  \[ \mathbb{E}[X_{1,i}S_{1,i}] = \mathbb{E}[X_{1,i}]\mathbb{E}[S_{1,i}] = \frac{1}{\mu \lambda_1} \]  
  \[ (4) \]

- The third term in (2) ($\mathbb{E}[X_{1,i}W_{1,i}]$): First we characterize the waiting time $W_{1,i}$ by means of two events $E_{1,i}^B$ and $E_{1,i}^L$ as
  \[ E_{1,i}^B = \{ T_{1,i-1} \geq X_{1,i} \} \],
  \[ E_{1,i}^L = \{ T_{1,i-1} < X_{1,i} \} \]  
  \[ (5) \]

- $T_{1,i} = S_{1,i} + W_{1,i}$, represents the system time of packet $1, i - 1$
Aol in a Multi-Source M/M/1 Queueing Model (2/9)

- **Event** $E_{1,i}^B$

- **Event** $E_{1,i}^L$

\[
W_{1,i} = \begin{cases} 
T_{1,i-1} - X_{1,i} + \sum_{i' \in \mathcal{M}_{2,i}^B} S_{2,i'}, & E_{1,i}^B \\
\sum_{i' \in \mathcal{M}_{2,i}^L} S_{2,i'}, & E_{1,i}^L
\end{cases}
\]  

- $\mathcal{M}_{2,i}^B$: the set of packets of source 2 that must be served under $E_{1,i}^B$
- $\mathcal{M}_{2,i}^L$: the set of packets of source 2 that must be served under $E_{1,i}^L$
Aol in a Multi-Source M/M/1 Queueing Model (3/9)

- The residual system time in the case \( E_{1,i}^B \)
  \[ R_{1,i}^B = T_{1,i-1} - X_{1,i} \]

- The total service time of source 2 packets in the case \( E_{1,i}^B \)
  \[ S_{1,i}^B = \sum_{i' \in M_{2,i}^B} S_{2,i'} \]

- The total service time of source 2 packets in the case \( E_{1,i}^L \)
  \[ S_{1,i}^L = \sum_{i' \in M_{2,i}^L} S_{2,i'} \]

- Consequently, the third term in (2) is given as
  \[
  \mathbb{E}[X_{1,i} W_{1,i}] = \left( \mathbb{E}[R_{1,i}^B X_{1,i} | E_{1,i}^B] + \mathbb{E}[S_{1,i}^B X_{1,i} | E_{1,i}^B] \right) P(E_{1,i}^B) 
  + \mathbb{E}[S_{1,i}^L X_{1,i} | E_{1,i}^L] P(E_{1,i}^L) 
  \] (7)
Aol in a Multi-Source M/M/1 Queueing Model (4/9)

- **Calculation of $P(E_{1,i}^B)$**

  $$P(E_{1,i}^B) = \int_0^\infty P(T_{1,i-1} \geq X_{1,i} | T_{1,i-1} = t) f_{T_{1,i-1}}(t) dt$$

  $= \int_0^\infty F_{X_{1,i}}(t) f_{T_{1,i-1}}(t) dt = 1 - \int_0^\infty e^{-\lambda_1 t} f_{T_{1,i-1}}(t) dt = 1 - L_T(\lambda_1), \quad (8)$$

- **$L_T(a)$** is given as

  $$L_T(\lambda_1) = \frac{(1 - \rho) \lambda_1 L_S(\lambda_1)}{\lambda_1 - \lambda(1 - L_S(\lambda_1))}, \quad (9)$$

- **$L_S(\lambda_1)$** is given as

  $$L_S(\lambda_1) = \int_0^\infty \mu e^{-(\mu + \lambda_1)s} ds = \frac{\mu}{\mu + \lambda_1}, \quad (10)$$

- **$E_{1,i}^L$** is the complimentary event of $E_{1,i}^B$

  $$P(E_{1,i}^L) = \frac{1 - \rho}{1 - \rho_2} \quad P(E_{1,i}^B) = 1 - P(E_{1,i}^L) = \frac{\rho_1}{1 - \rho_2}, \quad (11)$$
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**Aoi in a Multi-Source M/M/1 Queueing Model (5/9)**

- **Calculation of** $\mathbb{E}[R_{1,i}^B X_{1,i} | E_{1,i}^B]$ (The first term in (7))

$$
\mathbb{E}[R_{1,i}^B X_{1,i} | E_{1,i}^B] = \mathbb{E}[T_{1,i-1} X_{1,i} | E_{1,i}^B] - \mathbb{E}[X_{1,i}^2 | E_{1,i}^B] \tag{12}
$$

$$
= \int_0^{\infty} \int_0^{\infty} x t f_{X_{1,i}, T_{1,i-1} | E_{1,i}^B} (x, t) \, dx \, dt - \int_0^{\infty} x^2 f_{X_{1,i} | E_{1,i}^B} (x) \, dx,
$$

- **PDF** $f_{X_{1,i}, T_{1,i-1} | E_{1,i}^B} (x, t)$

$$
f_{X_{1,i}, T_{1,i-1} | E_{1,i}^B} = \begin{cases} 
0 & x > t \\
\mu^2 (1 - \rho)(1 - \rho_2) e^{-\lambda_1 x} e^{-\mu(1-\rho)t} & x \leq t 
\end{cases} \tag{13}
$$

- **PDF** $f_{X_{1,i} | E_{1,i}^B} (x)$

$$
f_{X_{1,i} | E_{1,i}^B} (x) = \mu (1 - \rho_2) e^{-\mu(1-\rho_2)x} \tag{14}
$$

- **The first term in (7)**

$$
\mathbb{E}[R_{1,i}^B X_{1,i} | E_{1,i}^B] = \frac{1}{\mu^2 (1 - \rho_2)(1 - \rho)} \tag{15}
$$
Calculation of $\mathbb{E}[S_{c,i}^B X_{c,i} | E_{c,i}^B]$ (The second term in (7))

$$
\mathbb{E}[S_{1,i}^B X_{1,i} | E_{1,i}^B] = \int_0^\infty x \mathbb{E}[\sum_{i' \in M_2^B} S_{2,i'} | E_{1,i}^B, X_{1,i} = x] f_{X_{1,i} | E_{1,i}^B}(x) dx
$$

$$
= \rho_2 \int_0^\infty x^2 f_{X_{1,i} | E_{1,i}^B}(x) dx
$$

PDF $f_{X_{1,i} | E_{1,i}^B}(x)$

$$
f_{X_{1,i} | E_{1,i}^B}(x) = \mu(1 - \rho_2)e^{-\mu(1-\rho_2)x}
$$

The second term in (7)

$$
\mathbb{E}[S_{1,i}^B X_{1,i} | E_{1,i}^B] = \frac{2\rho_2}{\mu^2(1 - \rho_2)^2}
$$
Calculation of $E[S_{1,i} X_{1,i} \mid E_{1,i}^L]$ (The third term in (7))

$$E[S_{1,i} X_{1,i} \mid E_{1,i}^L] = \int_0^\infty \int_0^\infty x E\left[ \sum_{i' \in M_{2,i}} S_{2,i'} \mid X_{1,i} = x, T_{1,i-1} = t, E_{1,i}^L \right] \cdots$$

$$f_{X_{1,i} T_{1,i-1} \mid E_{1,i}^L}(x, t) dx dt = \frac{1}{\mu} \int_0^\infty \int_0^\infty x E\left[ M_{2,i}^L \right] \cdots$$

$$X_{1,i} = x, T_{1,i-1} = t, E_{1,i}^L] f_{X_{1,i} T_{1,i-1} \mid E_{1,i}^L}(x, t) dx dt$$

$$= \frac{1}{\mu} \int_0^\infty \int_0^\infty x \sum_{m=0}^\infty m \Pr\left[ M_{2,i}^L = m \mid X_{1,i} = x, T_{1,i-1} = t, E_{1,i}^L \right] dx dt,$$

$$T_{1,i-1} = t, E_{1,i}^L] f_{X_{1,i} T_{1,i-1} \mid E_{1,i}^L}(x, t) dx dt,$$  (18)
The third term is given by

\[ \mathbb{E}[W_{1,i}X_{1,i}|E_{1,i}^L] = \frac{\lambda_1(1 - \rho)}{P(E_{1,i}^L)} \int_0^\infty \int_0^\infty (t + \tau)e^{-\mu(t + \rho_1\tau)} \cdots \]

\[
\left( \sum_{m=0}^\infty \sum_{j=0}^\infty m \bar{P}_{m|j}(\tau) \frac{\lambda_2 t^j}{j!} \right) d\tau dt \triangleq \frac{\lambda_1(1 - \rho)}{P(E_{1,i}^L)} \Psi(\mu, \rho_1, \lambda_2) \quad (19)
\]

The average AoI of source 1

\[ \Delta_1 = \lambda_1^2(1 - \rho)\Psi(\mu, \rho_1, \lambda_2) + \]

\[ \frac{1}{\mu} \left( \frac{1}{\rho_1} + \frac{\rho}{1 - \rho} + \frac{(2\rho_2 - 1)(\rho - 1)}{(1 - \rho_2)^2} + \frac{2\rho_1\rho_2(\rho - 1)}{(1 - \rho_2)^3} \right) \quad (20) \]
Results

- Exponential distribution with $\mu = 1$ and $\lambda_2 = 0.6$

![Graph showing results](attachment:image.png)


Thank You For Your Attention!

Questions?
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